I.M. Glazman

Direct Methods of Qualitative Spectral Analysis of Singular Differential Operators

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TABLE OF CONTENTS

PREFACE ix

INTRODUCTION 1

Chapter I. THE DECOMPOSITION METHOD AND GENERAL THEOREMS ON THE SPECTRUM 6

§1. Closed linear operators and self-adjoint operators in Hilbert space 6

1. Classification of the points of the spectrum of a closed linear operator 6
2. Extension spectra and decomposition of closed linear operators. . 10
3. Spectrum of a self-adjoint operator and trial manifolds 11
4. Relative complete continuity of symmetrical operators 17
5. Generalizations of H. Weyl's theorem on completely continuous perturbations 21
6. On the square of a closed symmetric operator with a finite defect number 24

§2. One-dimensional self-adjoint differential operators 27

7. Decomposition of operators and properties of the spectrum independent of the behavior of the coefficients at finite distances 27
8. Small and relatively small perturbations of differential operations 29
9. Test systems of finite functions and the continuous part of the spectrum of a singular operator as the set of accumulation points of the spectra of regular operators 32
10. The quadratic functional, and general criteria for the discreteness of the negative part of the spectrum 34
11. Localization principles. 37
12. The negative part of the spectrum and the oscillatory properties of differential operations 40
13. On differential operations on vector-functions 43
14. A finite-difference analog of the one-dimensional self-adjoint singular differential operator 44

§3. Multidimensional self-adjoint differential operators 47

15. Inclusion of multidimensional boundary-value problems in the general theory of symmetric operators 47
16. On the defect index of partial differential operators 54
17. Decomposition in multidimensional domains 59

18. Small and relatively small perturbations in the multi-dimensional case 63
19. The negative part of the spectrum and the quadratic functional . 65
20. S. L. Sobolev's spaces and the decomposition principle 70

§4. J-self-adjoint operators 74

21. The domain of values of a quadratic functional and the external regularity field of a linear operator 74
22. J-symmetric operators and their J-self-adjoint extensions . . . 76
23. Some spectral properties of J-self-adjoint extensions of J-symmetric operators with a finite defect number 77
24. J-symmetric differential operators 79

Chapter II. THE SPECTRUM OF ONE-DIMENSIONAL TWO-TERM DIFFERENTIAL OPERATORS 82

25. Two-term differential operations 82
26. Some auxiliary inequalities 83
27. Criteria for the boundedness and discreteness of the negative part of the spectrum. 85
28. Criteria for the semi-boundedness and discreteness of the whole spectrum 89
29. Spectrum discreteness criteria for the case of the polar differential operation 93
30. Criteria for non-oscillatory and oscillatory behavior 95
31. The positive part of the continuous spectrum 106
32. The spectral alternative for semi-bounded Schroedinger operators 113
33. The case when the spectrum of the Schroedinger operator covers the entire axis 115
34. Examples of the study of the character of the spectrum in the case of a one-term operation 118
35. On the spectrum of the Schroedinger operator with a complex-valued potential energy 121
36. The spectrum of two-term differential operators on vector functions 124
37. A discrete analog of the oscillation theorems 125

Chapter III. THE SPECTRUM OF ONE-DIMENSIONAL DIFFERENTIAL OPERATORS OF GENERAL FORM 129

38. One-dimensional differential operations of general form 129
39. Criteria for the semi-boundedness of the spectrum 129
40. Domains of non-oscillatory and oscillatory behavior 132
41. On gaps in the continuous part of the spectrum 134
42. Continuity criteria for the positive part of the spectrum 135

43. Birman's theorem 137
44. M. G. Krein's theorem on the multiplicative structure of positive differential operators 140

Chapter IV. THE SPECTRUM OF MULTIDIMENSIONAL DIFFERENTIAL OPERATORS 146

45. Multidimensional differential operations 146
46. Theorems on the discrete part of the spectrum 146
47. On criteria for the semi-boundedness and discreteness of the spectrum of multidimensional boundary-value problems 152
48. The negative part of the spectrum and the nodal lines of the solutions of elliptic equations 155
49. The character of the continuous part of the spectrum as related to the behavior of the potential and to the form of the infinite domain 159
50. On the discrete part of the spectrum of the Laplace operator in limit-cylindrical domains 167

Chapter V. THE CHARACTER OF THE SPECTRUM OF THE SCHROEDINGER DIFFERENTIAL OPERATOR AND THE BEHAVIOR OF THE SOLUTIONS OF THE CORRESPONDING HOMOGENEOUS EQUATION 175

51. Formulation of the problem 175
52. Some a priori estimates for the solution of the homogeneous elliptic equation 176
53. Behavior of the solutions at points of the discrete part of the spectrum 178
54. An estimate of the distance of a point from the spectrum by the behavior of the solution at this point 181
55. The behavior of the solutions at points of the continuous part of the spectrum 183
56. The spectrum of the Hill differential operator 187
57. The Hartman-Putnam theorem on the gaps in the spectrum of operators with a bounded potential 191
58. On V. A. Steklov's problem in the theory of orthogonal polynomials and its continual analog 195

Chapter VI. SOME ADDITIONAL PROBLEMS OF QUALITATIVE SPECTRAL ANALYSIS 199

59. Various generalizations of A. M. Molchanov's criterion. 199
60. The negative part of the spectrum of multidimensional two-term higher-order operators 201
61. Algebraic investigation of the spectrum of the energy operator of a harmonic oscillator 202

62. The Dirac operator 205
63. The Pauli operator 207
64. The spectrum of the energy operator of a many-particle system 209
65. The structure of the continuous part of the spectrum of the Schroedinger operator 211
66. Complex potentials 214
67. The model of the crystal lattice 216

LIST OF REFERENCES 220

INDEX 233

ANNOTATION

This monograph presents an account of the bases and development of direct methods for investigating the nature of the spectrum of singular differential operators. It bears upon periodical literature on the subject that has appeared over the last ten to fifteen years. The operator mainly considered is the Schroedinger operator.

The book should be of interest to mathematicians and theoretical physicists. Senior and post-graduate students may find it suitable for gaining an acquaintance with the qualitative spectral theory of differential equations.

PREFACE

This book primarily investigates the character of the spectrum of singular differential operators as dependent on the behavior of the coefficients of the corresponding differential operation. The singularity of the boundary-value problems considered in this way consists in the unboundedness of the domain of variation of the independent variables or in the fact that the coefficients of the differential operation have singularities on the boundary of this domain.

Although many various criteria of positions of parts of the spectrum are given in the book, it is nevertheless not intended for use as a collection of ready-made criteria. Its purpose is rather to supply means for obtaining criteria of this kind. These means, based on general propositions of the theory of operators, are called here direct methods of qualitative spectral analysis, and their application is illustrated by numerous examples. Many of them are connected with problems which appear in quantum mechanics.

In its initial form, the book was written exclusively on the basis of the author's papers /31/, but in its present form it gives many results obtained in recent years by other authors. A most significant part of the book is devoted to an account of a series of results obtained by M. Sh. Birman on the nature of the spectrum.

A detailed survey of the contents of the book as a whole and of the various chapters is given in the introduction.

I express deep gratitude to N. I. Akhiezer, M. Sh. Birman, and Yu. I. Lyubich, for it was due to contact with them that my work on the problems treated of in this book was set in motion in the first place. Their comments added greatly to the improvement of certain parts of the book. B. M. Levitan and V. A. Marchenko made some helpfully critical remarks; A. A. Egorov and M. N. Zaida assisted in reading the proofs. To all of them I owe sincere thanks.

I. M. Glazman

INTRODUCTION

The bases of the theory of singular differential operators were laid by Hermann Weyl in his famous paper "Über gewöhnliche Differentialgleichungen mit Singularitäten und die zugehörigen Entwicklungen wilkürlicher Funktionen" of 1910 /20 (2)/. Already in this classical work, along with the basic result of the expansion in the solutions of a singular Sturm-Liouville problem, there are certain theorems on the nature of the spectrum. These theorems came as the first results in the modern domain of investigations now known as qualitative spectral analysis.

Qualitative spectral analysis, or the investigation of the nature of the spectrum, consists in studying the set of points of the spectrum as dependent on the behavior of the coefficients of the differential operation, on the form of the domain and on the character of the boundary conditions.

After the fundamental work of H. Weyl almost no papers connected with singular boundary-value problems appeared in the course of fifteen years. The theory of singular differential operators owes its further progress to E. Schroedinger, who published in 1926 two short notes "Quantiesirung als Eigenwertproblem" /109/. Already in the first of these communications, which laid the mathematical foundation of quantum mechanics, Schroedinger obtained a stationary equation for the electron, known by his name.

Problems of determining the energy spectrum of concrete systems, which were studied after this in various works on quantum mechanics, were decisive for the further development of the theory of singular differential operators and, in particular, of qualitative spectral analysis.

Under the influence of the successes of quantum mechanics in the forties, and especially in the first post-war years, works on qualitative spectral analysis appeared regularly in the periodical mathematical literature, mainly abroad. Many various results of the investigation of the spectrum of the one-dimensional Schroedinger operator were obtained during 1946-1952 by A. Wintner, P. Hartman and K. Putnam.

Thus, in the 40 years which have elapsed since the publication of the fundamental paper of H. Weyl, through the efforts of many authors considerable material on qualitative spectral analysis of singular differential operators has been accumulated. This material, however, does not contain results concerning higher-order differential equations, and hardly deals (with the exception of the papers of K. Friedrichs and F. Rellich) with multidimensional singular boundary-value problems.

Further progress in this field of mathematical analysis was achieved largely thanks to the works of Soviet mathematicians. Since 1950 investigations on qualitative spectral analysis of higher-order differential operators and on the nature of the spectrum of multidimensional singular boundary-value problems have been published in the Soviet literature.

From a methodical point of view, one may distinguish two directions of research; one analytical and one connected with the theory of operators. Essentially, these directions were represented even earlier in the solution of the classical Sturm-Liouville problem on a finite interval, where to a certain extent they correspond to the asymptotic method of Liouville and to the variational method of Courant.

The analytic direction, included in the sphere of classical analysis, is based on asymptotic methods and on the apparatus of the theory of analytic functions. The two-volume monograph of E. C. Titchmarsh (/95(2)/, 1960; 1st edition—1946, 1958) and the well-known book of B. M. Levitan (/61(1)/, 1950) are devoted to analytic methods in the theory of singular second-order differential operators. Of the results regarding the nature of the spectrum of higher-order differential operators, obtained analytically, one should mention the theorems of I. M. Rapoport (/82(1)/, 1951) included in his monograph (/82(2)/, 1954), and some results of M. A. Naimark (see /72(3)/, 1954).

By their character the operator-theoretical methods are in a certain sense close to the direct methods of mathematical physics, and therefore they can be called direct methods of qualitative spectral analysis.

The basis of the direct methods of investigation of the nature of the spectrum are the decomposition method and the method of comparison of quadratic forms. These methods are based on the general theorems of the extension theory and on the spectral theory of operators in the Hilbert space, as well as on the well-known theorem of H. Weyl on the completely continuous perturbations. To the direct methods of qualitative spectral analysis of singular boundary-value problems belongs, in particular, the minimax principle of R. Courant, used in the case of unbounded domains by Courant himself (/54/, 1931), F. Rellich (/83 (4)/, 1948) and D. Jones (/40/, 1953). The development of the operator-theoretical methods of investigating the nature of the spectrum of singular differential operators was prepared by the works of R. Courant, K. Friedrichs and F. Rellich.

The object of investigation in the present book is the spectrum $S(L)$ of singular operators generated by various differential operations. The singularity of the operator L may be caused by singularities of the coefficients at a finite distance or by unboundedness of the domain of variation of the independent variables. Thus, for the Schroedinger operation in the case of a single electron, the singularities are the point where the nucleus is situated at the point at infinity. For the Tricomi operation in a bounded domain of the half-plane $y \geqslant 0$, the singular points fill that part of the axis $y = 0$ which belongs to the boundary of this domain. In the quantum-mechanical many-particle problem the manifolds of singular points extend to infinity.

The tendency to find a general basis for the investigation of the nature of the spectrum of one-dimensional singular differential operators of any order, of multidimensional singular boundary-value problems and of finite-difference operators led to the working out of operator-theoretical methods of qualitative spectral analysis and, in particular, of the decomposition method (see /31 (1)/, 1950/). This method made it possible in the investigation of certain properties of the spectrum to ignore the behavior of the coefficients of the differential operators outside the neighborhoods of the singular points, as well as to find out the influence exerted on

the spectrum by individual singular points or boundaries. The decomposition method is therefore in a certain sense a method for localizing singularities.

The basis of the decomposition method is the decomposition principle which, as applied to the simplest one-dimensional operation

$$-y''+q(x)y \qquad (L)$$

with one singular point at infinity consists in the following /31 (1)/. Let L be a self-adjoint operator generated in $\mathcal{L}_2(0,\infty)$ by the operation (L) together with some boundary conditions. By imposing on the functions from the domain of definition $\mathfrak{D}_L$ of the operator L for a given $\gamma>0$ the additional conditions $y(\gamma)=y'(\gamma)=0$, we obtain the manifold $\mathfrak{D}_{L^0}\subset\mathfrak{D}_L$. Then the operator L_0, naturally induced by the operator L on $\mathfrak{D}_{L_0}$, is the orthogonal sum $L_0=L_1\oplus L_\gamma$ of some symmetrical operators in $\mathcal{L}_2(0,\gamma)$ and $\mathcal{L}_2(\gamma,\infty)$. Taking self-adjoint extensions of these operators, we obtain the operator $M=\tilde{L}_1\oplus\tilde{L}_\gamma$.

Now, the given operator L and the constructed operator M are different self-adjoint extensions of the same "decomposed" operator L_0. Since these extensions are finite-dimensional and the operator $\tilde{L}_1$ is regular, it is natural to expect that certain properties of the spectrum which are caused by the singularity of the problem should be preserved when passing from L to $\tilde{L}_\gamma$.

These properties include the belonging of a given point λ to the continuous part $C(L)$ of the spectrum of the operator L, the clustering of the points of the spectrum towards the point $\lambda=-\infty$ and the existence of an infinite set of points of the spectrum in any one-sided neighborhood of the given point λ. The conservation of the first two properties can be written in the form of the relationship

$$C(L)=C(\tilde{L}_\gamma). \qquad (C)$$

if we agree to refer the point $\lambda=-\infty$ to $C(L)$ whenever it is a limit point of the spectrum $S(L)$ of the operator L. The conservation of the last property can be represented in the form of the relationship

$$S'(L)=S'(\tilde{L}_\gamma). \qquad (S')$$

where S' denotes the set of all right-hand limit points of the spectrum S'.

Relationship (C) expresses the above-mentioned decomposition principle, which may be extended to the multidimensional case as well (/31 (5)/, 1954), although in this latter case the operator L is already an infinite-dimensional extension of the decomposed operator L_0.

In addition to relationships (C) and (S'), in investigating the character of the spectrum by direct methods use is systematically made of test manifolds of finite functions (/31 (2)/, 1951). These manifolds are used in connection with a theorem following from spectral decomposition, by virtue of which the number of points of the spectrum of a self-adjoint operator A that lie between $\lambda_0-\delta$ and $\lambda_0+\delta$, or to the left of λ_0, is equal to the maximum dimension of the linear manifolds lying in $\mathfrak{D}_L$ on which the inequality $\|Af-\lambda_0 f\|<\delta\|f\|$, or correspondingly $(Af,f)<\lambda_0(f,f)$, is satisfied.

From this theorem it follows in particular that the continuous spectrum $C(A+B)$ of the sum of self-adjoint operators lies in the $\|B\|$-neighborhood of the continuous spectrum $C(A)$. On the additional assumption of the one-sidedness of the perturbation, when, for example, $B > 0$, the continuous spectrum $C(A+B)$ is situated in the right-hand $\|B\|$-neighborhood of $C(A)$.

The use of relationships (C),(S') and of the test manifolds, which are linear envelopes of infinite sequences of finite functions with disjoint supports going to infinity, is what constitutes the decomposition method. It is obvious that the decomposition method makes it possible in studying the singular properties of the spectrum to neglect the values of the coefficients of the differential operators at finite distance and to find the influence on the spectrum of some particular feature of their behavior at infinity. Both the order of the differential operator and the number of the independent variables become to a certain extent insignificant.

By means of the decomposition method numerous results were obtained regarding the nature of the spectrum of one-dimensional higher-order differential operators, and of multidimensional singular boundary-value problems, many of which proved to be new as compared with the results obtained earlier analytically, even for the case of the simplest operation (L). In particular, the decomposition method made it possible to observe a series of non-oscillation and oscillation criteria for differential operations not only of the second, but also of higher orders. The study of this problem is based on the investigation of the sign of the quadratic functional (Ly, y).

In 1952 A. M. Molchanov found a criterion for the discreteness of the spectrum of the Schroedinger operator with the potential bounded from below /70/. This criterion was obtained on the basis of a general lemma of F. Rellich (/83 (1)/, 1942), which reduces the problem of finding discreteness criteria for the spectrum of the operator L to the problem of the conditions for the complete continuity of the imbedding operator from the space with the metric (Ly, y) to the space with the metric (y, y).

A. M. Molchanov's result immediately drew attention by its completeness. This was the first necessary and sufficient criterion in qualitative spectral analysis. However, it was not at once noticed that not only the problem of the discreteness of the spectrum but also a number of other problems concerning the nature of the spectrum are connected with the complete continuity of certain imbedding operators.

This connection was observed by M. Sh. Birman, who in 1958 obtained on this basis a series of results which appreciably advanced the qualitative spectral theory of singular differential operators /13 (5)/. He worked along lines connected with the approach to problems of qualitative spectral analysis which, like the decomposition method, he developed for semi-bounded boundary-value problems.

M. Sh. Birman's method is the method of comparison of quadratic forms. He uses the theory of K. Friedrichs of semi-bounded operators in the Hilbert space H and considerably develops the methods indicated by this author in his fundamental investigation /97 (1)/, which was published as early as 1934. An important role in the application of the method of comparison of quadratic forms is played by M. Sh. Birman's generalization of the theorem of Weyl on the invariance of the continuous part of the spectrum under completely continuous perturbations to the case of perturbations which are completely continuous only relatively. If A is some positive

definite (possibly unclosed) operator in H, and K is some symmetric operator in H with the domain of definition $\mathfrak{D}_K \subset \mathfrak{D}_A$, then the operator K is said to be (relatively) completely continuous with respect to A when the operator $\mathring{A}^{-1}K$, where $\mathring{A}$ is Friedrich's extension of the operator A, is completely continuous in the metric (Af, f). When $K > 0$, relatively complete continuity is equivalent to complete continuity of the form (Kf, f) with respect to the form (Af, f).

Under certain limitations, we apply the method of comparison of quadratic forms, as well as the decomposition method, also to the investigation of the spectrum of unsymmetrical problems.

The key to the book is the short first chapter. The first section of this chapter may serve as a bridge over which the reader familiar with the general operator theory (see, for example /7/) can easily pass into the more specialized field that is the subject of the present investigation. In the following two sections an account and justification of the decomposition principle is given. Applications of the decomposition method for differential operators, as well as of the method of comparison of quadratic forms, to the obtainment of circumstantial results, are also given. These results indicate which variations should, or, conversely, cannot take place in the spectrum when the coefficients or the boundary conditions are varied in a given manner.

Thus, in contrast to the subsequent chapters, in Chapter I no special restrictions are imposed on the behavior of the coefficients of the differential operations at infinity. In particular, No. 12 clarifies the connection between spectrum and oscillatoriness. In the first reading it is possible to omit Nos. 6, 13, 14, 20 of the first chapter and the whole of the last section of the chapter, devoted to unsymmetric problems.

The second chapter is devoted to the application of the methods described in the first chapter to the obtainment of non-circumstantial results regarding the spectrum of a one-dimensional two-term $2n$-th-order differential operator which for $n = 1$ has the form (L). To non-circumstantial results belong affirmations about any properties of the spectrum for a given behavior of the coefficients of the differential operations at infinity.

The third and fourth chapters are devoted mainly to extending the results of the second chapter to one-dimensional differential operators of a general form and to multidimensional boundary-value problems.

The fifth chapter is devoted mainly to an investigation of the connection between the position of a point λ with respect to the spectrum of the Schroedinger operator and the behavior of the solutions of the corresponding homogeneous equation at infinity. In this chapter gap spectra are also studied. The fifth chapter essentially connects qualitative spectral analysis with certain problems of the qualitative theory of differential equations.

It also indicates the basis for the connection between direct and analytic methods of qualitative spectral analysis.

The sixth and last chapter is devoted to various additional problems and to more specialized ones, not included in the main text.

In each chapter an independent enumeration of the theorems and formulas is used, and in the first chapter an independent enumeration of the formulas is used in each section.

The list of references is given in [Cyrillic] alphabetical order of authors' names with internal enumeration of the works of each author.

An index of terms defined in the text follows the list of references.

Chapter I

THE DECOMPOSITION METHOD AND GENERAL THEOREMS ON THE SPECTRUM

§ 1. Closed linear operators and self-adjoint operators in Hilbert space

1. Classification of the points of the spectrum of a closed linear operator. In view of the absence of a commonly accepted terminology we begin with the definition of the basic concepts related to qualitative spectral analysis. The majority of these definitions may be found in /7/.

Let T be a closed linear operator with a domain of definition $\mathfrak{D}_T$, dense in the Hilbert space H.

The complex number λ is called a r e g u l a r - t y p e p o i n t of the operator T, if the following inequality is satisfied for all $f \in \mathfrak{D}_T$

$$\|(T-\lambda I)f\| > k_\lambda \|f\|, \tag{1}$$

where the number k_λ is positive and independent of f.

It follows from (1) that the regular-type points form an open set /7/. This set $\Pi(T)$ is called the r e g u l a r i t y f i e l d of the operator T. From the closure of the operator T and from inequality (1) it follows that for any point $\lambda \in \Pi(T)$ the manifold $(T-\lambda I)\mathfrak{D}_T$ is closed, i.e., it is a subspace.

If for a given λ of $\Pi(T)$ we have $(T-\lambda I)\mathfrak{D}_T = H$, then λ is called a r e g u l a r i t y p o i n t of the operator T. The totality of all the regularity points of the operator T is called its r e s o l v e n t s e t. This set is also open /7/.

The complementary set $S(T)$, of the resolvent set relative to the complex plane is called the s p e c t r u m of the operator T.

The spectrum of a closed linear operator is a closed set.

The set $D(T)$, of all eigenvalues of the operator T, is called the d i s c r e t e s p e c t r u m (or d i s c r e t e p a r t o f a s p e c t r u m) of the operator. The set of all eigenvectors, corresponding to a given value $\lambda \in D(T)$, is a subspace known as a p r o p e r s u b s p a c e, and its dimensionality is equal to the m u l t i p l i c i t y of the given eigenvalue.

When determining the number of points of the set $D(T)$ which lie in a given domain, each eigenvalue is to be counted a number of times equal to its multiplicity.

Suppose that for some element $f \neq 0$ of $\mathfrak{D}_T$ the equality

$$(T-\lambda I)^s f = 0 \tag{2}$$

holds for some integer s. Let σ be the smallest integer for which (2) is satisfied. We then have

$$g = (T-\lambda I)^{\sigma-1} f \neq 0, \quad Tg - \lambda g = 0,$$

so that $\lambda \in D(T)$.

The linear set of all the vectors $f \in \mathfrak{D}_T$, which satisfy equation (2) for a given $\lambda \in D(T)$ and some integer $s = s_f$, will be called the r o o t m a n i f o l d

/36/. A closed root manifold will be called a root subspace. Clearly a root subspace is invariant under the operator T.

The dimensionality of a root manifold, corresponding to a given value $\lambda \in D(T)$, is said to be the rank of the eigenvalue λ. Since a proper subspace is a part of a root manifold, the multiplicity of its eigenvalue cannot exceed its rank.

A root subspace G is called separable if there exists a subspace F, invariant under the operation T, in which it is continuously invertible and for which the direct sum of F and G gives the whole space H.

The set $C(T)$ of values of λ, for which there exists a bounded noncompact sequence $\{f_n\} \in \mathfrak{D}_T$, satisfying the condition

$$\lim_{n\to\infty} (T-\lambda I) f_n = 0 \tag{3}$$

is called the continuous spectrum (or continuous part of a spectrum) of the operator T. Any part of such a sequence, from which it is impossible to separate a convergent subsequence, is called characteristic of the given value $\lambda \in C(T)$.

The set $C(T)$ is also called the limit spectrum or concentration spectrum of the operator T; in this case the part of the set $C(T)$ remaining after the removal of the eigenvalues of the operator T is called the continuous spectrum. It will be shown below that the set $C(T)$ is closed.

The residual spectrum (or residual part of a spectrum) of the operator T is defined as the set $O(T)$ of values of λ for which the closure of the manifold $(T-\lambda I)\mathfrak{D}_T$ does not coincide with H and which do not belong to $D(T)$. Thus, the residual spectrum $O(T)$ is the set of all those points of $D(T^*)$ which do not belong to $D(T)$.

It is obvious that each of the sets $D(T)$, $C(T)$ and $O(T)$ belongs to the spectrum $S(T)$ of the operator T. Conversely it may be shown that each point of the set $S(T)$ belongs to at least one of the sets $D(T)$, $C(T)$, or $O(T)$.

Let $\lambda \in S(T)$. If $\lambda \in \Pi(T)$, then $\lambda \in O(T)$. If $\lambda \notin \Pi(T)$, then, according to (1), there exists a normalized sequence $\{f_n\} \subset \mathfrak{D}_T$ (i.e., a sequence normalized by the condition $\|f_n\| = 1$ for $n = 1, 2, \ldots$), such that (3) holds. If it is possible to separate from this sequence a convergent subsequence $f_{n_k} \to f$, then it follows from (3) that the element f belongs to $\mathfrak{D}_T$, and is an eigenvector of the operator T, with $\lambda \in D(T)$. If it is impossible to separate a convergent subsequence from the sequence $\{f_n\}$, then $\{f_n\}$ is noncompact, and, consequently, $\lambda \in C(T)$.

Thus, the spectrum $S(T)$ of the operator T is a set-theoretic sum of the discrete, continuous and residual parts of the spectrum

$$S(T) = D(T) \cup C(T) \cup O(T).$$

The components of the right-hand side of this equality may intersect. In particular, eigenvalues of infinite multiplicity always belong to the continuous part of the spectrum.

Note that the continuous part of the spectrum is not necessarily a perfect set, nor is the discrete part necessarily an isolated set. Thus, for example, the continuous part of the spectrum of any totally continuous operator consists of the single point $\lambda = 0$, and the residual part of the spectrum of any symmetrical, but not self-adjoint operator A with equal defect numbers contains an upper and lower λ-half plane, which are simultaneously contained in the discrete part of the spectrum of the adjoint operator A^*.

In the following we shall use the symbols $\oplus$ and $\ominus$ to denote the orthogonal sum and the orthogonal complement respectively of the linear manifolds of the space H. We shall use the symbol $\dotplus$ to denote a direct sum.

Theorem 1. The continuous spectrum $C(T)$ of a closed linear operator is a closed set.

Proof. It is sufficient to show that the complement of $C(T)$ to the whole plane is open.

Let $\lambda_0 \in C(T)$. If $\lambda_0 \in \Pi(T)$, then some neighborhood of the point λ_0 also belongs to $\Pi(T)$ and, consequently, does not intersect with $C(T)$. If $\lambda_0 \in \Pi(T)$, then the number λ_0 is an eigenvalue of finite multiplicity of the operator T. Let G be the proper subspace associated with the value $\lambda_0 \in D(T)$, and F' the intersection of $\mathfrak{D}_T$ with $H \ominus G$, so that

$$\mathfrak{D}_T = F' \oplus G.$$

The manifold F' is dense in $H \ominus G$ (see further, No. 6).

In a linear manifold F' the inequality

$$\|(T-\lambda_0 I)\varphi\| > k_{\lambda_0}\|\varphi\| \qquad (\varphi \in F'), \tag{4}$$

is satisfied for some $k_{\lambda_0} > 0$, since otherwise there would exist a normalized sequence $\{\varphi_n\} \subset F'$, for which

$$\lim_{n\to\infty} (T-\lambda_0 I)\varphi_n = 0.$$

But since $\lambda_0 \in C(T)$, the sequence $\{\varphi_n\}$ would be compact and therefore for some subsequence

$$\lim_{k\to\infty} \varphi_{n_k} = \varphi.$$

The element φ would then be an eigenvector of the operator T, corresponding to the eigenvalue λ_0; thus $\varphi \perp G$, which contradicts the definition of G.

From inequality (4), for all λ of some neighborhood of the point λ_0, follows the inequality

$$\|(T-\lambda I)\varphi\| > \tfrac{1}{2} k_{\lambda_0}\|\varphi\| \qquad (\varphi \in F'), \tag{5}$$

wherefrom $\lambda \in C(T)$.

Assuming that this were not so, we can find a noncompact normalized sequence $\{f_n\} \in \mathfrak{D}_T$, for which the limit relation (3) holds.

Denoting by φ_n and g_n the components of the vector f_n in F' and G respectively, we have

$$\begin{aligned} \|(T-\lambda I)\varphi_n\| &= \|(T-\lambda I)f_n - (T-\lambda I)g_n\| = \\ &= \|(T-\lambda I)f_n + (\lambda-\lambda_0)g_n\| \leqslant \|(T-\lambda I)f_n\| + |\lambda-\lambda_0|\,\|g_n\|. \end{aligned} \tag{6}$$

Further, since the sequence $\{f_n\}$ is noncompact, and $\dim G < \infty$, for some $\delta > 0$ and some sequence of numbers $n_k \to \infty$ we shall have $\|\varphi_{n_k}\| > \delta$, and, from 6, it follows that for λ sufficiently close to λ_0 we have the relation

$$\|(T-\lambda I)\varphi_{n_k}\| < \tfrac{1}{2} k_{\lambda_0}\delta,$$

which contradicts inequality (5). The theorem is thus proved.

Let $\lambda \in D(T)$ and let G be the corresponding proper subspace, and F' the intersection of the orthogonal complement to G with $\mathfrak{D}_T$. The concentration of T in F' will be denoted by T' (see /7/) and for $\lambda \in D(T)$ we assume $T' = T$.

It is then obvious that for any value λ the equality

$$(T-\lambda I)\mathfrak{D}_T=(T'-\lambda I)F' \tag{7}$$

holds and the operator $(T'-\lambda I)^{-1}$ exists.

Theorem 2. If $\lambda\in C(T)$, the operator $(T'-\lambda I)^{-1}$ is bounded. Conversely, if the operator $(T'-\lambda I)^{-1}$ is bounded and λ is not an eigenvalue of infinite multiplicity, then $\lambda\in C(T)$.

Proof. If $\lambda\in C(T)$, then, from inequality (4) it follows that the operator $(T'-\lambda I)^{-1}$ is bounded. Now if the operator $(T'-\lambda I)^{-1}$ is bounded and $\dim O<\infty$, then, as in the proof of Theorem 1, we obtain $\lambda\in C(T)$.

Theorem 3. If $\lambda\in C(T)$, then the manifold $(T-\lambda I)\mathfrak{D}_T$ is closed. Conversely, if this manifold is closed, and the point λ is not an eigenvalue of infinite multiplicity, then $\lambda\in C(T)$.

Proof. The first part of the theorem follows from the fact that the operator $(T'-\lambda I)^{-1}$ is bounded, from the closure of T and from equality (7).

Considering the second part of the theorem, we shall show that the operator $R'=(T'-\lambda I)^{-1}$ is bounded. For this purpose we continue linearly the closed operator R' with the domain of definition $\Delta=(T-\lambda I)\mathfrak{D}_T$ into the whole space H; R' is assumed to be zero on the orthogonal complement to Δ. The resulting operator R, being a closed linear operator defined in the entire space, is bounded according to the theorem of S. Banach /9/, and, therefore, the operator R' is also bounded.

From the above definitions it follows directly that every component of the spectrum of the orthogonal sum $T_1\oplus T_2$ of the operators T_1 and T_2 is equal to the set-theoretic sum of the corresponding components of the spectrum of these operators

$$D(T_1\oplus T_2)=D(T_1)\cup D(T_2),$$
$$C(T_1\oplus T_2)=C(T_1)\cup C(T_2),$$
$$O(T_1\oplus T_2)=O(T_1)\cup O(T_2)$$

and, consequently, also that

$$S(T_1\oplus T_2)=S(T_1)\cup S(T_2).$$

In certain special problems (see No. 65) it is necessary to separate from the continuous spectrum $C(A)$ of a self-adjoint operator A the so-called absolutely continuous part of the spectrum, which is defined as follows /55/.

The set of elements $g\in H$, for which the non-decreasing function $(E_\lambda g, g)$ is absolutely continuous, forms some subspace $H_C(A)$, invariant with respect to A /80/. The part A_C of the operator A which acts in $H_C(A)$ is called the absolutely continuous part of the operator A, and the spectrum $S(A_C)$ of this operator A_C is called the absolutely continuous part of the spectrum of the operator A. It is obvious that $S(A_C)\subset C(A)$. We note that the subspace $H_D(A)=H\ominus H_C(A)$ consists of all those elements $g\in H$, for which the function $(E_\lambda g, g)$ is singular. It is natural to call the part A_D of the operator A, acting in $H_D(A)$, the singular part of the operator A, and its spectrum $S(A_D)$ the singular part of the spectrum of the operator A. Evidently, $D(A)\subset S(A_D)$.

The eigenvalues of $S(A_D)$ form the discrete component of the singular part of the spectrum of the operator A. The complement of this component relative to the entire spectrum $S(A_D)$ will be called the continuous component of the singular part of the spectrum of the operator A.

2. **Extension spectra and decomposition of closed linear operators.** From the definition of the spectrum and its components it follows that if $\tilde{T}$ is a closed linear extension of the closed operator T, then

$$C(\tilde{T}) \supset C(T), \quad D(\tilde{T}) \supset D(T),$$

but

$$O(\tilde{T}) \subset O(T).$$

We shall call the extension $\tilde{T}$ finite dimensional, if $\dim \mathfrak{D}_{\tilde{T}} < \infty$ $(\mathrm{mod}\, \mathfrak{D}_T)$, and infinite dimensional, if the opposite is the case.

Theorem 4. If $\tilde{T}$ is a finite-dimensional linear extension of the closed operator T, then $C(\tilde{T}) = C(T)$.

The proof is complete on the basis of the previous theorem. The proof of Theorem 4, which is independent of the particular theorem of S. Banach, is given below.

It is sufficient to show that if $\lambda \notin C(T)$, then $\lambda \notin C(\tilde{T})$. Let $\lambda \notin C(T)$. If $\lambda \in D(\tilde{T})$, then the multiplicity of the eigenvalue λ is finite, since it does not exceed $k+m$, where k is the multiplicity of the number λ which is the eigenvalue of the operator T, and $m = \dim \mathfrak{D}_{\tilde{T}}$ $(\mathrm{mod}\, \mathfrak{D}_T)$. Now, on the strength of Theorem 2, it is sufficient to show that the operator $(\tilde{T}' - \lambda I)^{-1}$ is bounded.

Let $\tilde{G}$ be the proper subspace of the operator $\tilde{T}$, corresponding to the eigenvalue λ (if $\lambda \notin D(\tilde{T})$, then we set $\tilde{G} = 0$), and let $\tilde{P}$ be the projection operator onto $H \ominus \tilde{G}$.

If $h \in (\tilde{T} - \lambda I)\mathfrak{D}_{\tilde{T}}$, then $h = f + g$, where f is the projection of h on $\Delta = (T - \lambda I)\mathfrak{D}_T$, and g belongs to some finite-dimensional subspace F, which is orthogonal to Δ.

We have

$$\begin{aligned} \|(\tilde{T}' - \lambda I)^{-1} h\| &= \|(\tilde{T}' - \lambda I)^{-1} f + (\tilde{T}' - \lambda I)^{-1} g\| = \\ &= \|\tilde{P}(T' - \lambda I)^{-1} f + (\tilde{T}' - \lambda I)^{-1} g\| \leqslant \\ &\leqslant \|(T' - \lambda I)^{-1} f\| + \|(\tilde{T}' - \lambda I)^{-1} g\| \leqslant \\ &\leqslant M\|f\| + \mu\|g\| = (M + \mu)\|h\|, \end{aligned}$$

so that the operator $(\tilde{T}' - \lambda I)^{-1}$ is bounded.

In the above we have used the boundedness of the operator $(T' - \lambda I)^{-1}$ in the finite-dimensional subspace F, as well as the boundedness of the operator $(\tilde{T}' - \lambda I)^{-1}$ in this subspace.

From Theorem 4 it follows that the continuous spectra of various finite-dimensional extensions of a closed linear operator coincide.

The representation of a closed linear operator as an extension of an orthogonal sum of two other closed operators is called the decomposition of the operator. A decomposition is said to be finite-dimensional if this extension is finite dimensional. The decomposition of the operator T is defined by the equalities

$$T = \tilde{T}_0, \quad T_0 = T_1 \oplus T_2,$$

and from Theorem 4 for finite dimensional extensions follows the relation

$$C(T) = C(T_1) \cup C(T_2). \tag{8}$$

which expresses the principle of decomposition for closed operators. This principle will be further extended in No. 17 to a certain infinite dimensional extensions.

3. Spectrum of a self-adjoint operator and trial manifolds. If the operator A is self-adjoint, then $O(A)=\emptyset$, and, consequently,

$$S(A)=D(A)\cup C(A).$$

Since any non-real value λ is a regularity point of the operator A, the spectrum $S(A)$ of a self-adjoint operator is a closed set of the points of the real axis. As is known, the spectrum $S(A)$ coincides with the set of the growth points of the spectral function E_λ of the operator A. The part $D(A)$ of the spectrum $S(A)$ then coincides with the set of discontinuity points of the monotonic operator-function E_λ and, consequently, is either denumerable or finite.

Theorem 5. If λ_0 is a non-isolated growth point of the function E_λ, then $\lambda_0\in C(A)$. Conversely, if the point λ_0 belongs to $C(A)$ but is not an eigenvalue of infinite multiplicity, then λ_0 is a non-isolated growth point of the function E_λ.

Proof. If λ_0 is a non-isolated growth point of the function E_λ, then in each neighborhood of the point λ_0, an interval Δ may be found which does not contain λ_0 and is such that $E(\Delta)\neq\emptyset$. (The symbol D will denote the empty set.) Thus, there exists an infinite sequence of non-intersecting intervals such that $E(\Delta_n)\neq\emptyset$. $\lambda_0\bar{\in}\Delta_n$, the ends of the interval Δ_n converging to λ_0.

Choosing arbitrarily in each subspace $E(\Delta_n)H$ a normalized vector f_n, we obtain an infinite orthonormalized and, consequently, non-compact sequence for which we have

$$\lim_{n\to\infty}(A-\lambda_0 I)f_n=0,$$

so that $\lambda_0\in C(A)$.

If the point λ_0 belongs to $C(A)$, but is not an eigenvalue of infinite multiplicity, then, according to Theorem 2, the operator $(A'-\lambda_0 I)^{-1}$ is not bounded. On the other hand, if λ_0 were an isolated growth point of the function E_λ, then it would also be a constancy point of the spectral function E'_λ of the operator A' in the closure of F' and, consequently, the operator $(A'-\lambda_0 I)^{-1}$ would be bounded, which proves the theorem.

If the operator A a priori cannot have eigenvalues of infinite multiplicity (for example, one-dimensional differential operators) then it follows from what was stated above that $C(A)$ coincides with the set of non-isolated points of the spectrum $S(A)$.

The theorems which follow relate certain properties of the spectrum with the existence in $\mathfrak{D}_A$ of trial manifolds. In the formulation of Theorem 6, and onwards the dimensionality of the orthogonal complement $H\ominus F$ is called the defect number of the subspace F and is denoted by $\mathrm{Def}\,F$.

Theorem 6 /31 (2)/. A necessary and sufficient condition for the interval $\Delta=(\lambda_0-\delta,\ \lambda_0+\delta)$ to contain only a finite set of the points of the spectrum of a self-adjoint operator A is that there exist a subspace F with a finite defect number, such that for all $f\in\mathfrak{D}_A\cap F$ the following inequality is satisfied:

$$\|(A-\lambda_0 I)f\|>\delta\|f\|. \tag{9}$$

The proof follows from the spectral formula

$$\|(A-\lambda_0 I)f\|^2=\int_{-\infty}^{\infty}|\lambda-\lambda_0|^2\,d(E_\lambda f,\ f). \tag{10}$$

In fact, assuming that the intersection $S(A)\cap\Delta$ consists of a finite number of eigenvalues of finite multiplicity and denoting by G the linear envelope of all the eigenvectors which correspond to the eigenvalues of Δ, we obtain with the aid of (10) the following relation for the elements $f\in\mathfrak{D}_A$ orthogonal to G:

$$\|(A-\lambda_0 I)f\|^2=\int\limits_{|\lambda-\lambda_0|>\delta}|\lambda-\lambda_0|^2\,d(E_\lambda f,\ f)>\delta^2\|f\|^2, \qquad (11)$$

that is, inequality (9).

Conversely, if (9) holds for all $f\in\mathfrak{D}_A\cap F$, then assuming that the set $S(A)\cap\Delta$ is not exhausted by a finite number of eigenvalues of multiplicity, we find that the manifold $E(\Delta)H$ is infinite dimensional. Now, if we take in $E(\Delta)H$ an element $f\in\mathfrak{D}_A\cap F$, for example any vector of $E(\Delta)H$ which is orthogonal to the projection of G on $E(\Delta)H$, then we obtain from (10)

$$\|(A-\lambda_0 I)f\|^2=\int\limits_{\lambda_0-\delta}^{\lambda_0+\delta}|\lambda-\lambda_0|^2\,d(E_\lambda f,\ f)$$

leading to the inequality

$$\|(A-\lambda_0 I)f\|\leqslant\delta\|f\|,$$

where $f\in\mathfrak{D}_A\cap F$, contradicting (9).

These considerations establish the following theorem, which is more precise than the previous one.

Theorem 6^{bis}. *A necessary and sufficient condition for the number of points of the spectrum of a self-adjoint operator A lying in a given interval $\Delta=(\lambda_0-\delta,\ \lambda_0+\delta)$ not to exceed the number m is that there exists a subspace F with a defect number m, such that inequality (9) is satisfied for all $f\in\mathfrak{D}_A\cap F$.*

Theorem 6 gives the necessary and sufficient condition for the absence of points of the continuous part of the spectrum in the given interval. From this theorem follows, in particular,

Theorem 7 /31 (2)/. If the interval $\Delta=(\lambda_0-\delta,\ \lambda_0+\delta)$ does not contain the continuous part of the spectrum of a self-adjoint operator A, and B is a bounded, self-adjoint operator with a norm smaller than δ, then the point λ_0 does not belong to the continuous part of the spectrum of the sum $A+B$.

From inequality (9) for $f\in\mathfrak{D}_A\cap F$ we have

$$\|(A+B-\lambda_0 I)f\|\geqslant\|(A-\lambda_0 I)f\|-\|Bf\|>(\delta-\|B\|)\|f\|.$$

For one-sided perturbations the above theorem allows the following refinement.

Theorem 7^{bis}. *If A is a self-adjoint operator, for which $C(A)\cap(\alpha,\ \beta)=\emptyset$, and B is a non-negative operator with a norm $\|B\|=\gamma<\beta-\alpha$ then $C(A+B)\cap(\alpha+\gamma,\ \beta)=\emptyset$.*

Proof. From the relation $C(A)\cap(\alpha,\ \beta)=\emptyset$ we obviously have the relation $C\left(A+\frac{\gamma}{2}I\right)\cap\left(\alpha+\frac{\gamma}{2},\ \beta+\frac{\gamma}{2}\right)=\emptyset$, so that for a given λ of the interval $(\alpha+\gamma,\ \beta)$ there exists, according to Theorem 6, a subspace F with a finite defect number, such that

$$\left\|\left(A+\frac{\gamma}{2}I-\lambda I\right)f\right\|>\frac{\gamma+\delta}{2}\|f\| \qquad (\delta>0)$$

with $f \in \mathfrak{D}_A \cap F$. From this relation and from the equality

$$\left\| B - \tfrac{\gamma}{2} I \right\| = \tfrac{\gamma}{2}$$

it follows that for any $f \in \mathfrak{D}_A \cap F$ we will have the inequality

$$\|(A+B-\lambda I)f\| \geqslant \left\|\left(A+\tfrac{\gamma}{2}I-\lambda I\right)f\right\| - \left\|\left(B-\tfrac{\gamma}{2}I\right)f\right\| > \delta\|f\|.$$

From the above inequality together with Theorem 6, we obtain the following solution

$$C(A+B) \cap (\alpha+\gamma, \beta) = \emptyset,$$

which was what we set out to prove.

If $\alpha = -\infty$, but $\gamma < \infty$, the above considerations are still valid. However, if $\gamma = \infty$ and $\alpha = -\infty$, the above proof and the statement itself are no longer valid.

From Theorem 7 follows the well-known theorem of H. Weyl:

Theorem 8 /20 (1)/. If we add to a self-adjoint operator A a totally continuous self-adjoint operator K, then the continuous part of the spectrum $A+K$ is not changed, so that $C(A+K) = C(A)$.

Proof. Let $\lambda_0 \in C(A+K)$, but $\lambda_0 \notin C(A)$, so that for some $\delta > 0$ the relation $(\lambda_0 - \delta, \lambda_0 + \delta) \cap C(A) = \emptyset$ holds. Representing the operator K in the form $K = K_1 + K_2$, where K_1 is a bounded finite dimensional operator and K_2 is a self-adjoint operator having a small norm $\|K_2\| < \delta$, we obtain by Theorems 6 and 7 the relation $\lambda_0 \notin C(A+K)$, which contradicts the original assumption. From the symmetry of the formulation of the theorem with respect to the operators A and $A+K$, Theorem 8 is proved. Some generalizations of this theorem will be dealt with in No. 5.

Theorem 9 /31 (3)/. A necessary and sufficient condition for $\Delta = (\lambda_0 - \delta, \lambda_0 + \delta)$ to contain a finite set of points of the spectrum of a self-adjoint operator A, is that there exists an infinite dimensional linear manifold $O \subset \mathfrak{D}_A$, for which

$$\|(A - \lambda_0 I)f\| < \delta\|f\|. \tag{12}$$

Proof. That the condition is necessary is obvious, if we take $O = E(\Delta)H$.

To prove that the condition is sufficient we shall assume that, contrary to this theorem, the interval Δ contains only a finite set of eigenvalues, and denote by F the orthogonal complement of the subspace $E(\Delta)H$ to the entire space H. The subspace F has a finite defect number, and for $f \in F$ inequality (11) will be satisfied.

Since the subspace $E(\Delta)H$ is finite dimensional, it is possible to find in O a vector f which is orthogonal to $E(\Delta)H$. For this vector inequality (11) holds, thus contradicting (12).

From these considerations we have established the following more precise theorem:

Theorem 9bis. The number of points of the spectrum of a self-adjoint operator A, lying in the interval $\Delta = (\lambda_0 - \delta, \lambda_0 + \delta)$, is equal to the maximum dimensionality of the linear manifolds $O \subset \mathfrak{D}_A$, for which inequality (12) is satisfied.

Theorem 9 gives a necessary and sufficient condition for the presence of points of the continuous part of the spectrum in a given interval.

Another criterion for the existence of the continuous part of the spectrum in a given interval is given by the following theorem, which can easily be established by spectral decomposition. This theorem is used in Chapter V.

Theorem 10. A necessary and sufficient condition for the interval $\Delta=(\lambda_0-\delta, \lambda_0+\delta)$ *to contain an infinite set of points of the spectrum of a self-adjoint operator* A, *is that there exists a non-compact bounded sequence* $\varphi_n \in \mathfrak{D}_A$, *for which*

$$\|A\varphi_n-\lambda_0\varphi_n\|<\delta\|\varphi_n\| \qquad (n=1, 2, \ldots).$$

In all that follows (with the exception of No. 6) the symmetrical operators are assumed to have equal defect numbers. The defect number of a symmetrical operator A is denoted by Def A.

Theorem 9 gives rise to the following proposition, which in the case of symmetrical operators is a refinement of Theorem 4 which has already been proved.

Theorem 11. Let A *be a symmetrical operator with a finite defect number. If the set of points of the spectrum of one of the self-adjoint extensions of* A, *lying in a given interval* (α, β), *is infinite, then the set of points of the spectrum of any other self-adjoint extension of this operator lying in the same interval is also infinite.*

Obviously, each point λ of the continuous part of the spectrum of a self-adjoint extension $\tilde{A}$ of the operator A belongs to one of the following four types.

1) for some $\delta_1>0$, and any $\delta_2>0$

$$(\lambda-\delta_1, \lambda)\cap S(\tilde{A})=\emptyset, \quad (\lambda, \lambda+\delta_2)\cap S(\tilde{A})\neq\emptyset,$$

2) for any $\delta_1>0$ and some $\delta_2>0$

$$(\lambda-\delta_1, \lambda)\cap S(\tilde{A})\neq\emptyset, \quad (\lambda, \lambda+\delta_2)\cap S(\tilde{A})=\emptyset,$$

3) for any $\delta_1>0$ and any $\delta_2>0$

$$(\lambda-\delta_1, \lambda)\cap S(\tilde{A})\neq\emptyset, \quad (\lambda, \lambda+\delta_2)\cap S(\tilde{A})\neq\emptyset,$$

4) for some $\delta_1>0$ and some $\delta_2>0$

$$(\lambda-\delta_1, \lambda)\cap S(\tilde{A})=\emptyset, \quad (\lambda, \lambda+\delta_2)\cap S(\tilde{A})=\emptyset.$$

From Theorem 11 it follows that the type of each point of the continuous part of the spectrum of a self-adjoint extension $\tilde{A}$ of a given symmetrical operator A with a finite defect number does not depend on the choice of the extension.

The gap of the continuous part of the spectrum of a symmetrical operator A is defined as any adjacent interval of the closed set $C(A)$.

By means of the lemma of Heine-Borel and on the basis of the property of analyticity of the resolvent it is shown in /53 (2)/ that if the interval (α, β) belongs to the resolvent set of a symmetrical operator with a finite defect number, then the spectrum of any self-adjoint extension $\tilde{A}$ of the operator A in the interval (α, β) consists of an isolated set of eigenvalues of finite multiplicity. The following proposition, which ensues directly from Theorem 10, contains and, in its second part, somewhat complements, this result.

Theorem 11[bis] *If the interval* (α, β) *is the gap of the continuous part of the spectrum* $C(A)$ *of a symmetrical operator with a finite defect number, then this interval is the gap of the continuous part of the spectrum* $C(A)$ *of any self-adjoint extension* $\tilde{A}$ *of the operator* A.

If for some self-adjoint extension of A *the endpoint* α *(or* β*) of the interval is a limit point of the set of eigenvalues lying inside the gap* (α, β), *then for any self-adjoint extension* $\tilde{A}$ *this same point* α *(or respectively,* β*) of the interval will also be a limit point of the set of eigenvalues lying inside the gap* (α, β).

The following theorems are established in a similar way to Theorems 6 (6[bis]) and 9 (9[bis]). The interval $(\lambda_0 - \delta\ \lambda_0 + \delta)$ then passes to the semi-axis $(-\infty, \lambda_0)$.

Theorem 12. In order that the part of the spectrum of a self-adjoint operator A, *lying to the left of a given point* λ_0, *be a finite set, it is necessary and sufficient that there exists a subspace* F *with a finite defect number, such that for all* $f \in \mathfrak{D}_A \cap F$ *the following inequality is satisfied:*

$$(Af - \lambda_0 f,\ f) > 0 \qquad (13)$$

Theorem 12[bis]. *A necessary and sufficient condition for the number of points of the spectrum of a self-adjoint operator* A, *lying to the left of a given point* λ_0, *not to exceed the number* m, *is that there exists a subspace* F *with a defect number* m, *such that for all* $f \in \mathfrak{D}_A \cap F$ *inequality (13) is satisfied.*

Theorem 13 /31 (3)/. A necessary and sufficient condition for the part of the spectrum of a self-adjoint operator A, *lying to the left of a given point* λ_0, *to be an infinite set, is that there exists an infinite dimensional manifold* $G \subset \mathfrak{D}_A$ *for which*

$$(Af - \lambda_0 f,\ f) < 0. \qquad (14)$$

Theorem 13[bis]. *The number of points of the spectrum of a self-adjoint operator* A, *lying to the left of a given point* λ_0, *is equal to the maximum dimension of the linear manifolds* $G \subset \mathfrak{D}_A$ *for which inequality (14) is satisfied.*

If for any $\lambda < \lambda_0$ the intersection $(-\infty, \lambda) \cap S(A)$ contains a finite number of eigenvalues of finite multiplicity, then it is said that to the left of the point λ_0 the spectrum is discrete. The latter obviously implies that the part of the spectrum $S(A)$ lying in the interval $[-\infty, \lambda_0]$ does not have other limit points except, possibly, the point λ_0.

From Theorem 13 the next theorem directly follows:

Theorem 14 /53 (2)/. If the set of points, lying to the left (right) of the point λ_0, *of the spectrum of one of the self-adjoint extensions of a symmetrical operator with a finite defect number is infinite, then the set of points of the spectrum of any other self-adjoint extension of this operator, lying to the left (or, respectively, to the right) of the point* λ_0, *is also infinite.*

In particular, it follows from Theorem 14 that *if the spectrum of one of the self-adjoint extensions of a given symmetrical operator with a finite defect number is bounded from below (above), then the spectrum of any other self-adjoint extension of this operator is also bounded from below (above).* The last statement remains true for some symmetrical operators with an infinite defect number (see, further, No. 19).

With respect to further applications, we note that in the case of a functional Hilbert space it is sufficient, by virtue of Theorem 12, to verify inequality (13) only on real functions. In fact, if inequality (13) is satisfied on the real envelope of two real functions φ and ψ, then

$$|(A\varphi - \lambda_0\varphi,\ \psi)|^2 \leqslant (A\varphi - \lambda_0\varphi,\ \varphi)(A\psi - \lambda_0\psi,\ \psi)$$

and, consequently,

$$([A - \lambda_0 I](\varphi + i\psi),\ \varphi + i\psi) \geqslant [\sqrt{(A\varphi - \lambda_0\varphi,\ \varphi)} - \sqrt{(A\psi - \lambda_0\psi,\ \psi)}]^2 \geqslant 0.$$

We may also note that if inequality (9) is satisfied on the linear real envelope of two real functions φ and ψ, then for the function $\varphi + i\psi$ we shall have

$$\|(A - \lambda_0 I)(\varphi + i\psi)\| > \frac{\delta}{\sqrt{2}}\|\varphi + i\psi\|.$$

Let the real value λ_0 be a regular-type point of a symmetrical operator A. If Def $A < \infty$, then for any self-adjoint extension $\tilde{A}$ of the operator A we have $\lambda_0 \overline{\in} C(A)$, however for certain extensions of the operator A it may be seen that $\lambda_0 \in D(\tilde{A})$ (see /7/). On the other hand, if Def $A = \infty$, then for some self-adjoint extensions $\tilde{A}$ we may have not only $\lambda_0 \in D(\tilde{A})$, but even $\lambda_0 \in C(\tilde{A})$. However, in all the above cases, the intersection of the spectra $S(\tilde{A})$ of all the self-adjoint extensions $\tilde{A}$ of the operator A does not contain regular-type points of the operator A. The respective theorem formulated below belongs to J. Calkin /56/, and the proof given here is taken mainly from the note of M. I. Vishik /24 (1)/.

Theorem 15 /56/. *If λ_0 is a regular-type point of a symmetrical operator A with a finite defect number then there exists a self-adjoint extension $\tilde{A}$ of the operator A, for which $\lambda_0 \overline{\in} S(\tilde{A})$.*

Proof. Without restricting generality, we set $\lambda_0 = 0$ and denote by G the space of the solutions of the equation $A^*g = 0$. Also, let the linear manifold $\mathfrak{M}$ be the complete inverse image of the subspace G when mapping by means of the operator A^*, so that

$$A^*\mathfrak{M} = G.$$

Since $A^*G = 0$, then $G \subset \mathfrak{M}$ and the linear manifold $\mathfrak{M}$ can be represented as an orthogonal sum

$$\mathfrak{M} = G \oplus \mathfrak{N},$$

where $\mathfrak{N}$ is some linear manifold.

Let the operator $\tilde{A} \subset A^*$ be defined in the domain

$$\mathfrak{D}_{\tilde{A}} = \mathfrak{D}_A \dotplus \mathfrak{N}; \tag{15}$$

we shall show that the operator $\tilde{A}$ is self-adjoint and that the point λ_0 does not belong to its spectrum.

First we note that the sum in (15) is direct since from the equality

$$\varphi + h = 0 \qquad (\varphi \in \mathfrak{D}_A,\ h \in \mathfrak{N})$$

we have, after applying the operator A^*, the equality $A\varphi + A^*h = 0$, whence, by virtue of the orthogonality of the components, we obtain $A\varphi = A^*h = 0$; but zero is a regular-type point of the operator A, so that $\varphi = 0$ and, consequently, $h = 0$.

If $f \in \mathfrak{D}_{\tilde{A}}$, $g \in \mathfrak{D}_{\tilde{A}}$ and

$$f = \varphi + u,$$
$$g = \psi + v,$$

where φ and ψ belong to $\mathfrak{D}_A$, and u and v belong to $\mathfrak{N}$, the symmetry of the operator $\tilde{A}$ follows from the relations

$$(\tilde{A}f,\ g) = (\tilde{A}[\varphi + u],\ \psi + v) = (A\varphi,\ \psi) + (A\varphi,\ v) + (\tilde{A}u,\ \psi),$$
$$(f,\ \tilde{A}g) = (\varphi + u,\ \tilde{A}[\psi + v]) = (\varphi,\ A\psi) + (\varphi,\ \tilde{A}v) + (u,\ A\psi).$$

Finally, from (15) we have

$$\tilde{A}\mathfrak{D}_{\tilde{A}} = A\mathfrak{D}_A \oplus A^*\mathfrak{N} = A\mathfrak{D}_A \oplus G = H,$$

so that the operator $\tilde{A}$ is self-adjoint and the value $\lambda = 0$ is its regularity point, which proves the theorem.

4. Relative complete continuity of symmetrical operators. We start by defining a more precise terminology for semi-bounded operators.

A symmetrical operator A is called s e m i - b o u n d e d, if for some $\gamma > -\infty$ and for all $f \in \mathfrak{D}_A$ the inequality

$$(Af, f) \geqslant \gamma (f, f). \tag{16}$$

is satisfied. The largest value α of the constant γ, for which inequality (16) is still satisfied for all $f \in \mathfrak{D}_A$, is called the l o w e r b o u n d of the operator A. If $\alpha > 0$, the operator A is called p o s i t i v e d e f i n i t e. If $\alpha \geqslant 0$, the operator A is called n o n - n e g a t i v e.

A non-negative operator A satisfies the Cauchy-Bunyakovskii inequality

$$|(Af, g)|^2 \leqslant (Af, f)(Ag, g),$$

from which it follows that

$$(Af, f) = 0$$

is possible only for $Af = 0$.

A non-negative operator A will be called p o s i t i v e if it satisfies the following inequality

$$(Af, f) > 0,$$

for all non-vanishing elements of $f \in \mathfrak{D}_A$.

By means of any positive operator A it is possible to define a scalar product in the linear manifold $\mathfrak{D}_A$ as follows:

$$[f, g]_A = (Af, g). \tag{17}$$

The $A\sim$ metric defined by formula (17) transforms $\mathfrak{D}_A$ into some (generally speaking, incomplete) Hilbert space.

If the operator A is positive definite, then the $A\sim$ metric transforms $\mathfrak{D}_A$ into such a linear metrized system H_A which may be completed by elements of H. The complete Hilbert space so obtained will be denoted by H_A (see /69, (1)/). The scalar product $[f, g]_A$, initially defined by formula (17) only on $\mathfrak{D}_A$, is additionally defined by continuity for any f and g of H_A.

As was established by K. Friedrichs /97 (1)/, for any symmetrical positive definite operator A there exists a self-adjoint extension $\mathring{A}$ for which

$$H_{\mathring{A}} = H_A. \tag{18}$$

It follows from the above relation that the lower bounds of the operators A and $\mathring{A}$ coincide. K. Friedrichs used this fact to prove the assumption of G. Neumann /73/ on the possibility of a self-adjoint extension of a semi-bounded operator preserving the lower bound. M.G. Krein has shown /53 (2)/ that the self-adjoint extension $\mathring{A}$ of the operator A is uniquely determined by condition (18). According to M.G. Krein, the operator $\mathring{A}$ is called the s t r i c t extension of the operator A. It is clear that the relation $\mathfrak{D}_{\mathring{A}} \subset \mathfrak{D}_A \subset H_A \subset H$ holds for the manifolds under consideration. Replacing A by $\mathring{A}$ is the right-hand side of formula (17) for the scalar product $[f, g]_A$ in H_A, the formula will then hold for any $f \in \mathfrak{D}_{\mathring{A}}$ and $g \in H_A$. We also note that the space H_A coincides with $\mathfrak{D}_{\mathring{A}^{\frac{1}{2}}}$ and for any $f, g \in H_A$, the right-hand side of (17) should be replaced by $\left(\mathring{A}^{\frac{1}{2}} f, \mathring{A}^{\frac{1}{2}} g\right)$ /53 (2)/.

If the operator A is semi-bounded from below, then for some $\mu > 0$ the operator $A + \mu I$ will obviously be positive definite. The metrics defined in

by the formula

$$[f, g]=([A+\mu I]f, g) \tag{19}$$

are obviously topologically equivalent for the various μ for which the operator $A+\mu I$ is positive definite, and lead after closure of $\mathfrak{D}_A$ to the same space $H_A \subset H$. These metrics will be called the A-metrics.

If for a given μ the operator $A+\mu I$ is positive, but it is not known whether it is positive definite, then the metric defined by formula (19) will be called an $A\sim$ metric (tilde instead of a hyphen).

If for a given μ the operator $A+\mu I$ is positive, but not positive definite, then the $A\sim$ metric, corresponding to this value of μ is in general not equivalent to the A-metrics, and the closure of $\mathfrak{D}_A$ in the $A\sim$ metric may be impossible with the elements of the H space.

The terms A- and $A\sim$ convergence, A- and $A\sim$ boundedness, A- and $A\sim$ density, and so on, may be defined in a similar manner to the definition of the terms A-, and $A\sim$ metric.

Let A be a positive definite (possibly not closed) operator with a domain of definition $\mathfrak{D}_A$ dense in H, and let K be some symmetrical (generally not closed) operator with a domain of definition $\mathfrak{D}_K=\mathfrak{D}_A$. We shall say that the operator K is completely continuous relative to A if the operator $\mathring{A}^{-1}K$ is completely continuous in H_A.

If $A=I$ (or, more generally, when the operator A is bounded), the relative complete continuity coincides with the ordinary complete continuity. It can be easily seen that from the complete continuity of the operator K in the usual sense follows its complete continuity relative to any positive definite operator A.

The operator $\mathring{A}^{-1}K$ is symmetrical in H_A, since the form

$$[\mathring{A}^{-1}Kf, f]_A=(Kf, f) \tag{20}$$

is real in $\mathfrak{D}_A$. If the operator $\mathring{A}^{-1}K$ is completely continuous in H_A, its extension by continuity to the whole of H_A will also be completely continuous. This extension will be denoted by T.

If the operator K is positive, then the complete continuity of K relative to A is equivalent to the compactness in the $K\sim$ metric of any set of $\mathfrak{D}_A$, bounded in the A-metric. This follows from the simple Theorem 16.

Theorem 16 /14 (5). A necessary and sufficient condition for a positive operator K to be completely continuous relative to a positive definite operator A ($\mathfrak{D}_K=\mathfrak{D}_A$), is that any A-bounded set of elements of $\mathfrak{D}_A$ is K-compact.

Proof. Let the operator K be completely continuous relative to A, so that the operator $\mathring{A}^{-1}K$ is completely continuous in H_A, and let $\{f_n\}$ be an arbitrary A-bounded sequence of elements of $\mathfrak{D}_A$.

Since the operator $\mathring{A}^{-1}K$ is completely continuous in H_A, it is possible to isolate from the sequence $\{\mathring{A}^{-1}Kf_n\}$ an A-convergent sequence $\{\mathring{A}^{-1}Kf_{n_k}\}$. But then the subsequence $\{f_{n_k}\}$ is $K\sim$ convergent, since

$$(K[f_{n_k}-f_{n_r}], [f_{n_k}-f_{n_r}])=[\mathring{A}^{-1}K(f_{n_k}-f_{n_r}), (f_{n_k}-f_{n_r})]_A.$$

Conversely, let any A-bounded set of elements of $\mathfrak{D}_A$ be $K\sim$ compact.

Then, in view of (20), the quadratic functional (Kf, f), as well as the quadratic functional $[\mathring{A}^{-1}Kf, f]_A$ will be bounded in H_A. The operator $\mathring{A}^{-1}K$ will therefore be A-bounded in $\mathfrak{D}_A$, and therefore it can be extended by

continuity to the whole H_A, by T. Since $T>0$, it is possible to define in $\mathfrak{D}_A$ a symmetrical operator $\sqrt{\mathring{A}^{-1}K}$ as follows: $\sqrt{\mathring{A}^{-1}K}f=+\sqrt{T}f$ for $f\in\mathfrak{D}_A$.

Let $\{f_n\}$ be any A-bounded sequence of H_A, and $\{\varphi_n\}$—a sequence of $\mathfrak{D}_A$, satisfying in the metric of H_A the condition

$$\lim_{n\to\infty}\{f_n-\varphi_n\}=0. \tag{21}$$

If the sequence $\{\varphi_{n_k}\}$ is the K-convergent part of the sequence $\{\varphi_n\}$ then it follows from (20) that the sequence $\{\sqrt{T}\varphi_{n_k}\}$ and, consequently, in view of (21), the sequence $\{\sqrt{T}f_{n_k}\}$ also, will be A-convergent. Thus the operator $\sqrt{T}$ is completely continuous in H_A. Consequently the operator T and a fortiori the operator $\mathring{A}^{-1}K\subset T$, are completely continuous in H_A.

The following theorem, which does not assume that the operator K is positive, gives a criterion for relative complete continuity which is useful in applications of the theory. It shows that the existence of a positive majorant of the operator K, completely continuous relative to A, implies complete continuity of the operator K itself relative to A.

Theorem 17 /13 (5)/. Let a symmetrical operator K and a positive operator K_1 be defined on $\mathfrak{D}_A$ and suppose that the inequality

$$|(Kf,\ f)|\leqslant(K_1f,\ f)$$

is satisfied for all $f\in\mathfrak{D}_A$. If the operator K_1 is completely continuous relative to the operator A then the operator K will also be completely continuous relative to A.

Proof. For any $f\in\mathfrak{D}_A$ we have

$$\left|[\mathring{A}^{-1}Kf,\ f]_A\right|=|(Kf,\ f)|\leqslant(K_1f,\ f)=[\mathring{A}^{-1}K_1f,\ f]_A. \tag{22}$$

Since the operator $\mathring{A}^{-1}K_1$ is completely continuous in H_A it is therefore bounded there. From (22) it also follows that the operator $\mathring{A}^{-1}K$ which is symmetrical in H_A is bounded in the metric of H_A. Since the bounded symmetrical operators $\mathring{A}^{-1}K$ and $\mathring{A}^{-1}K_1$ are defined on the manifold $\mathfrak{D}_A$, A-dense in H_A, they can be extended by continuity to the whole H_A. Let these extensions be denoted by T and T_1 respectively. The inequality

$$|[Tf,\ f]_A|\leqslant[T_1f,\ f]. \tag{23}$$

which is true on $\mathfrak{D}_A$, is still true for the whole of H_A.

It remains to show that the complete continuity in H_A of the self-adjoint operator T follows from (23) and from the complete continuity in H_A of the self-adjoint operator T_1.

Separating H_A into an orthogonal sum of positive and negative spectral subspaces of the operator T, we obtain

$$H_A=F\oplus G,\quad T=TP+TQ, \tag{24}$$

where P and Q are projection operators in H_A onto F and G respectively.

To complete the proof we have to show the complete continuity of the operators TP and TQ in H_A.

Since the operator T_1 is completely continuous in H_A it follows that the operator PT_1P is also completely continuous in H_A and, consequently, so is the operator $\sqrt{PT_1P}$.

Replacing f by Pf in (23) we obtain

$$[PTPf,\ f]_A\leqslant[PT_1Pf,\ f]_A$$

or, taking positive radicals,

$$[\sqrt{PTP}\, f,\ \sqrt{PTP}\, f]_A \leqslant [\sqrt{PT_1P}\, f,\ \sqrt{PT_1P}\, f]_A. \qquad (25)$$

The complete continuity in H_A of the operator $\sqrt{PTP}$ as well as of the operator $PTP = TP$, follows from (25) and from the complete continuity of the operator $\sqrt{PT_1P}$ in H_A.

Similarly it follows that the operator $QTQ = TQ$ is completely continuous in H_A. Therefore, according to (24), the operator T is completely continuous in H_A and the theorem is proved.

To conclude we prove a lemma which will be used in No. 5.

Lemma 1 /13 (5)/. If A is a positive definite operator, and K a symmetrical operator completely continuous relative to A, then the operator

$$B = A + K$$

is semi-bounded from below and the B-metric is equivalent to the A-metric.

Proof. Since the extension T of the operator $A^{-1}K$ to the whole of H_A is completely continuous in H_A, we represent the operator T in the form

$$T = R + S,$$

where R is some bounded finite dimensional operator in H_A, and the norm of the operator S in H_A does not exceed a given number $\varepsilon > 0$.

If $\{\varphi_k\}$ is a complete orthonormalized system of the eigenvectors of the self-adjoint operator T in H_A, and $\{\lambda_k\}$ is the corresponding sequence of eigenvalues, then

$$Rf = \sum_{k=1}^{n} \lambda_k [f,\ \varphi_k]_A \varphi_k,$$

where n is chosen so that $|\lambda_k| < \varepsilon$ for $k > n$. With this choice of n and for any $f \in H_A$ we have

$$|[Sf,\ f]_A| \leqslant \sum_{k=n+1}^{\infty} |\lambda_k|\, |[f,\ \varphi_k]_A|^2 < \varepsilon [f,\ f]_A.$$

The elements φ_k may not belong to $\mathfrak{D}_A$, but replacing them by elements $\psi_k \in \mathfrak{D}_A$ nearby in the A-metric we obtain

$$R = R_1 + R_2,$$

where

$$R_1 f = \sum_{k=1}^{n} \lambda_k [f,\ \psi_k]_A \psi_k,$$

and the norm of R_2 in H_A does not exceed ε.

Now

$$T = R_1 + S_1,$$

where $S_1 = R_2 + S$, and, consequently, the norm of S_1 in H_A does not exceed 2ε.

Furthermore, for any $f \in \mathfrak{D}_A$ we have

$$[R_1 f,\ f]_A = \sum_{k=1}^{n} \lambda_k [f,\ \psi_k]_A [\psi_k,\ f]_A = \sum_{k=1}^{n} \lambda_k (Af,\ \psi_k)\,(A\psi_k,\ f) = \sum_{k=1}^{n} \lambda_k (f,\ A\psi_k)\,(A\psi_k,\ f),$$

so that

$$|[R_1 f,\ f]_A| \leqslant \sum_{k=1}^{n} |\lambda_k| \cdot \|A\psi_k\|^2 \cdot \|f\|^2 < M \|f\|^2,$$

and since

$$(Kf, f)=[Tf, f]_A=[R_1f, f]_A+[S_1f, f]_A,$$

we obtain the inequality

$$|(Kf, f)|\leqslant M\|f\|^2+2\varepsilon(Af, f).$$

This inequality leads to the relation

$$(Bf, f)=(Af, f)+(Kf, f)\geqslant(Af, f)-M\|f\|^2-2\varepsilon(Af, f),$$

therefore

$$(Bf, f)+M\|f\|^2\geqslant(1-2\varepsilon)(Af, f).$$

Taking $\varepsilon<\frac{1}{2}$, inequality (19) shows that the operator B is semi-bounded from below. Therefore the A-convergence follows from the B-convergence.

On the other hand, the operator T, being completely continuous in H_A, is bounded there. Denoting its norm in H_A by $\|T\|_A$, we have

$$(Bf, f)=(Af, f)+(ATf, f)\leqslant(1+\|T\|_A)(Af, f),$$

therefore, in view of (16)

$$(Bf, f)+M\|f\|^2\leqslant\left(1+\|T\|_A+\frac{M}{\alpha}\right)(Af, f),$$

where α is the lower bound of the operator A. Thus, the B-convergence follows from the A-convergence and the lemma is proved.

5. Generalizations of H. Weyl's theorem on completely continuous perturbations. In this subsection we shall generalize the theorem of H. Weyl in two different ways. Firstly we extend the theorem proved by H. Weyl for self-adjoint operators to the case of any closed operators (see also /36/). Then we give the generalization due to M.Sh. Birman /13 (5)/ of H. Weyl's theorem to the case of relative completely continuous perturbations.

The first of these generalizations can be conveniently made with the aid of the following lemma, which will be used again in No. 24.

Lemma 2. From any noncompact bounded sequence $\{f_n\}$ of elements of a Hilbert space, it is possible to isolate a subspace $\{f_{n_k}\}$ such that the sequence of the differences $\{f_{n_{k+1}}-f_{n_k}\}$ is also noncompact.

Proof. Since the sequence $\{f_n\}$ is noncompact, then, by virtue of Hausdorff's compactness cirterion, for some $\varepsilon>0$ there does not exist in H a finite set of elements such that the whole sequence $\{f_n\}$ lies in its ε-neighborhood. Thus it is possible to isolate from the given sequence a subsequence such that the distance between any two elements of the subsequence is not smaller than ε. This subsequence we denote by $\{f_n^{(1)}\}$.

Furthermore, using the weak compactness of the sphere of the Hilbert space, we isolate from the subsequence $\{f_n^{(1)}\}$ the weakly convergent part $\{f_n^{(2)}\}$. It is obvious that for any $g\in H$ we then have

$$\lim_{n\to\infty}(f_{n+1}^{(2)}-f_n^{(2)}, g)=0. \tag{26}$$

We now show that $\{f_n^{(2)}\}$ is the required subsequence.

Assuming this not to be so, we choose in the sequence $\{f_{n+1}^{(2)}-f_n^{(2)}\}$ the convergent part $\{f_{2n}^{(3)}-f_{2n-1}^{(3)}\}$. Assuming

$$h=\lim_{n\to\infty}(f_{2n}^{(3)}-f_{2n-1}^{(3)}),$$

we obtain, in view of (26),

$$\|h\|^2 = \lim_{n\to\infty} (f_{2n}^{(3)} - f_{2n-1}^{(3)},\ h) = 0,$$

so that

$$\lim_{n\to\infty} (f_{2n}^{(3)} - f_{2n-1}^{(3)}) = 0, \tag{27}$$

contradicting the inequality

$$\|f_{2n}^{(3)} - f_{2n-1}^{(3)}\| \geqslant \varepsilon > 0,$$

which holds by construction. Thus the lemma is proved.

The above proof of Lemma 2 was proposed to the author by M.I. Kadets.

Theorem 18. Adding a completely continuous operator K to a closed linear operator T does not alter the continuous part of the spectrum, i.e.,

$$C(T+K) = C(T).$$

Proof. It is sufficient to show that from $\lambda \in C(T)$ it follows that $\lambda \in C(T+K)$. Let $\lambda \in C(T)$ and let $\{f_n\}$ be a normalized compact sequence of $\mathfrak{D}_T$, for which

$$\lim_{n\to\infty} (Tf_n - \lambda f_n) = 0.$$

Let us isolate from the bounded sequence $\{f_n\}$ the weakly convergent subsequence $\{f_n^{(1)}\}$. The completely continuous operator K transforms this subsequence into a strongly convergent sequence $\{Kf_n^{(1)}\}$.

Now, using Lemma 2, we isolate from the subsequence $\{f_n^{(1)}\}$ the part $\{f_n^{(2)}\}$, such that the sequence of the differences $\{f_{n+1}^{(2)} - f_n^{(2)}\}$ is noncompact.

Setting $h_n = f_{n+1}^{(2)} - f_n^{(2)}$, we obtain a noncompact bounded sequence $\{h_n\}$ of $\mathfrak{D}_{T+K}$, for which

$$\lim_{n\to\infty} Kh_n = 0,$$

and, consequently,

$$\lim_{n\to\infty} [(T+K)h_n - \lambda h_n] = \lim_{n\to\infty} (Th_n - \lambda h_n) = 0,$$

from which it follows that $\lambda \in C(T+K)$, thus proving the theorem.

The second generalization of H. Weyl's theorem, indicated at the beginning of the subsection, leads to the following theorem.

Theorem 19 /13 (5)/. If a symmetric operator A is positive definite, and the symmetric operator K completely continuous relative to A, then the operator

$$B = A + K$$

is bounded from below and

$$C(\mathring{B}) = C(\mathring{A}).$$

Proof. In Lemma 1 of No. 4 it is proved that the operator B is bounded from below. Without loss of generality B may be considered positive definite.

Furthermore the same lemma shows that the A- and B-metrics are equivalent. Thus the real quadratic functional $(Bf, f) - (Af, f)$ is bounded in H_A; therefore there exists in H_A a bounded self-adjoint operator T such that

$$[f, g]_B - [f, g]_A = [f, Tg]_A \tag{28}$$

for all $f \in H_A$.

For $f \in \mathfrak{D}_A$ we have

$$[f, g]_B = (Bf, g) = (\mathring{A}\mathring{A}^{-1}Bf, g) = [\mathring{A}^{-1}Bf, g]_A$$

and relation (28) gives

$$\mathring{A}^{-1}Bf - f = Tf,$$

or

$$\mathring{A}^{-1}(B - A)f = Tf,$$

so that for $f \in \mathfrak{D}_A$

$$Tf = \mathring{A}^{-1}Kf \ ; \tag{29}$$

therefore the operator T is completely continuous in H_A.

Setting

$$\varphi = \mathring{B}^{-1}h, \quad \psi = \mathring{A}^{-1}h,$$

for any $h \in H$, we obtain for all $f \in H_A$

$$[f, \varphi]_B = [f, \mathring{B}^{-1}h]_B = (f, h) = (f, \mathring{A}\psi) = [f, \psi]_A.$$

This relation together with (28) gives

$$[f, \psi]_A - [f, \varphi]_A = [f, T\varphi]_A,$$

that is

$$[f, \mathring{A}^{-1}h - \mathring{B}^{-1}h]_A = [f, T\varphi]_A,$$

thus

$$\mathring{A}^{-1}h - \mathring{B}^{-1}h = T\mathring{B}^{-1}h \qquad (h \in H). \tag{30}$$

To prove the theorem it remains to show the complete continuity of the operator $T\mathring{B}^{-1}$ in H.

Let the set $F \subset H$ be bounded in H. Then the set $\mathring{B}^{-1}F$ is B-bounded, which implies that it is also A-bounded. Consequently, the set $T\mathring{B}^{-1}F$ is A-compact, and a fortiori compact in H. Thus, the operator $T\mathring{B}^{-1}$ in (30) is completely continuous in H. From the theorem of H. Weyl we have $C(\mathring{B}^{-1}) = C(\mathring{A}^{-1})$, and, consequently, also $C(\mathring{B}) = C(\mathring{A})$.

From Lemma 1 and from the above theorem it follows, in particular, that the spectrum of the operator $\mathring{B}$ to the left of the first point of the continuous part of the spectrum of the operator $\mathring{A}$, is discrete and bounded from below. This was shown for the first time by K. Friedrichs /97 (1)/.

In one case (see No. 43) it is possible to use together with Theorem 19 the following similar theorem.

Theorem 20 /13 (5)/. Let the metrics resulting from two positive definite operators A and B with domains of definition $\mathfrak{D}_B = \mathfrak{D}_A$ be equivalent (so that $H_A = H_B$) and $K = B - A$. Furthermore let K_1 be a positive operator with a domain of definition $\mathfrak{D}_{K_1} = \mathfrak{D}_A$, such that the quadratic functional $(K_1 f, f)$ is A-bounded, and $\tilde{K}_1$ is the extension by A-continuity of K_1 to the whole H_A.

If for all $f \in \mathfrak{D}_A$ we have

$$|(Kf, f)| \leqslant (K_1 f, f) \tag{31}$$

and if the boundedness of the set $F \subset H$ implies that the set $\mathring{B}^{-1}F$ is $\tilde{K}_1$-compact, then

$$C(\mathring{B}) = C(\mathring{A}).$$

Proof. Since the quadratic functional $(K_1 f, f)$ is A-bounded, then for any $f \in \mathfrak{D}_A$

$$(K_1 f, f) \leqslant M(Af, f). \tag{32}$$

In combination with (31), this inequality leads, for any $f, g \in \mathfrak{D}_A$, to the inequality

$$(Kf, g)^2 \leqslant M^2 (Af, f)(Ag, g).$$

By virtue of this relation, the bilinear functional (Kf, g) together with the operator K, can be extended by A-continuity to the whole H_A. This extension will be denoted by $\tilde{K}$. It is then obvious that for any $f \in H_A$ we shall have

$$|(\tilde{K}f, f)| \leqslant (\tilde{K}_1 f, f)$$

i.e., for any $f, g \in H_A$

$$|(\tilde{K}f, g)|^2 \leqslant (\tilde{K}_1 f, f)(\tilde{K}_1 g, g).$$

For $g = \mathring{A}^{-1}\tilde{K}f$ the last inequality gives

$$(\mathring{A}g, g)^2 \leqslant (\tilde{K}_1 f, f)(\tilde{K}_1 g, g),$$

which together with (32) leads to the relation

$$(\mathring{A}g, g)^2 \leqslant M(\tilde{K}_1 f, f)(\mathring{A}g, g)$$

or

$$(\mathring{A}g, g) \leqslant M(\tilde{K}_1 f, f). \tag{33}$$

On the other hand, as in the proof of the previous theorem, we obtain relations (28), (29) and (30). Thus, from (28) we obtain the following equality for any $f \in H_A$

$$Tf = \mathring{A}^{-1}\tilde{K}f,$$

since the sequence of the vectors $\mathring{A}^{-1}\varphi_n$ converges in H_A, provided the sequence $\varphi_n \in H_A$ is A-convergent.

Let F be a bounded set of elements $h \in H$. Then according to the conditions of the theorem the set $f = \mathring{B}^{-1}h$ $(h \in F)$ is $\tilde{K}_1$-compact, therefore, from (33), it follows that the set of elements $g = T\mathring{B}^{-1}h$ $(h \in F)$ is compact in H_A and a fortiori, also in H.

Thus, a set F bounded in H is transformed by the operator $T\mathring{B}^{-1}$ into a set compact in H, i.e., the operator $T\mathring{B}^{-1}$ is completely continuous in H. This means that the difference (30) of the operators $\mathring{A}^{-1}$ and $\mathring{B}^{-1}$ is completely continuous. Therefore, according to H. Weyl's theorem, the continuous parts of the spectra of the operators $\mathring{A}$ and $\mathring{B}$ coincide, thus proving the theorem.

6. On the square of a closed symmetric operator with a finite defect number. By introducing the square of a symmetric operator with a finite defect number it may be shown that Theorems 6 and 12, 9 and 13, 11 and 14 may be paired off so that each member of a given pair is the consequent of the other.

Let us show, for example, how to derive Theorem 11 from Theorem 14. We assume, for the time being without proof, that the domain of definition of the operator A^2 is dense in H, and that its defect number is finite if A is a symmetric operator with a finite defect number.

Passing to the proof of Theorem 11, we assume that in the interval $(\lambda_0 - \delta, \lambda_0 + \delta)$ there is an infinite set of points of the spectrum of some self-adjoint extension $\tilde{A}_1$ of the operator A. Hence with the aid of the

spectral formula we find that the negative part of the spectrum of the self-adjoint operator $B_1 = (\widetilde{A}_1 - \lambda_0 I)^2 - \delta^2 I$ is an infinite set. Then by virtue of Theorem 14, for any other self-adjoint extension $\widetilde{A}_2$ of the operator A, the negative part of the spectrum of the self-adjoint operator $B_2 = (\widetilde{A}_2 - \lambda_0 I)^2 - \delta^2 I$ will also be an infinite set, since both operators B_1 and B_2 are self-adjoint extensions of the same symmetric operator $(A - \lambda_0 I)^2 - \delta^2 I$ having a finite defect number. Since the set of points of the negative part of the spectrum of the operator B_2 is infinite in the interval $(\lambda_0 - \delta, \lambda_0 + \delta)$, there is an infinite set of points of the spectrum of the operator $\widetilde{A}_2$ and Theorem 11 is therefore proved.

To show that the assumptions made with respect to the operator A^2 are valid, we give some lemmas established in /31 (1)/ in connection with other problems. The generalization of these propositions to Banach spaces is given in /5/ and /35/.

Lemma 3. A linear manifold $\mathfrak{M}$, dense in the space H, is at the same time dense in any subspace F having a finite defect number. That is, the closure of the intersection $\mathfrak{M} \cap F$ coincides with F.

Proof. Let $\operatorname{Def} F = m$ and let $\{g_r\}_1^m$ be the base of the space $H \ominus F$.

Since the manifold $\mathfrak{M}$ is dense in H, then for any $f \in F$ there exists a sequence $\{\varphi^{(k)}\} \subset \mathfrak{M}$ of vectors, such that

$$\lim_{k \to \infty} \varphi^{(k)} = f. \tag{34}$$

To prove the theorem, it is necessary to find a similar sequence of vectors $\mathfrak{M} \cap F$. For this purpose we choose in the manifold $\mathfrak{M}$ the vectors $\widetilde{g}_1, \widetilde{g}_2, \ldots, \widetilde{g}_m$ which are so near to the vectors $g_1, g_2, \ldots, g_m$ respectively, that the determinant $\operatorname{Det} \|(\widetilde{g}_j, g_k)\|_{j,k=1}^m$ is nonvanishing. We now pass from each vector $\varphi^{(k)}$ to the vector

$$\psi^{(k)} = \alpha_1^{(k)} \widetilde{g}_1 + \alpha_2^{(k)} \widetilde{g}_2 + \ldots + \alpha_m^{(k)} \widetilde{g}_m + \varphi^{(k)}.$$

The numbers $\alpha_1^{(k)}, \alpha_2^{(k)}, \ldots, \alpha_m^{(k)}$ are so chosen that the vector $\psi^{(k)}$ is situated in the subspace F, for which these numbers can be determined from the following system of equations

$$\sum_{j=1}^{m} (\widetilde{g}_j, g_r)\, \alpha_j^{(k)} + (\varphi^{(k)}, g) = 0 \qquad (r = 1, 2, \ldots, m).$$

Since from condition (34) it follows that

$$\lim_{n \to \infty} \alpha_j^{(n)} = 0 \qquad (j = 1, 2, \ldots, m),$$

then the sequence $\mathfrak{M} \cap F$ belonging to the intersection $\{\psi^{(k)}\}$ converges to the vector f and Lemma 3 is proved.

Lemma 4. If the zero belongs to the regularity field $\Pi(T)$ of a closed linear operator T with a domain of definition dense in H and $\operatorname{Def} T\mathfrak{D}_T < \infty$, and $F \subset H$ is a subspace with a finite defect number, then the manifold $T(\mathfrak{D}_T \cap F)$ is a subspace and

$$\operatorname{Def} T(\mathfrak{D}_T \cap F) = \operatorname{Def} T\mathfrak{D}_T + \operatorname{Def} F.$$

Proof. The closure of the linear manifold $T(\mathfrak{D}_T \cap F)$ follows from the boundedness of the operator T^{-1}, and from the closure of the operator T. Let

$$\operatorname{Def} T\mathfrak{D}_T = m_1, \quad \operatorname{Def} F = m_2$$

and choose in the subspace $H \ominus T\mathfrak{D}_T$ the base $\{g_k\}_1^{m_1}$, and in the subspace $H \ominus F$ the base $\{f_k\}_1^{m_2}$.

Let us first show that

$$\operatorname{Def} T(\mathfrak{D}_T \cap F) \geqslant m_1 + m_2. \tag{35}$$

To prove this inequality it is sufficient to find $m_1 + m_2$ linearly independent vectors which are orthogonal to the subspace $T(\mathfrak{D}_T \cap F)$. As the first m_1 of these vectors we may take $g_1, g_2, \ldots, g_{m_1}$, since $T(\mathfrak{D}_T \cap F) \subset T\mathfrak{D}_T$.

To construct the remaining m_2 vectors, we extend the linearly bounded operator T^{-1} with the domain of definition $T\mathfrak{D}_T$ to the whole space H, and denote the resulting operator by R. The adjoint operator R^* will also be defined in the whole of H. In particular, it will be defined in the subspace $H \ominus F$.

Consider the vectors

$$R^* f_1, R^* f_2, \ldots, R^* f_{m_2}.$$

These m_2 vectors are orthogonal to the subspace $T(\mathfrak{D}_T \cap F)$, since for any $\varphi \in \mathfrak{D}_T \cap F$ we have

$$(T\varphi, R^* f_k) = (RT\varphi, f_k) = (\varphi, f_k) = 0 \qquad (k = 1, 2, \ldots m_2).$$

It remains to show the linear independence of the $m_1 + m_2$ vectors

$$g_1, g_2, \ldots, g_{m_1}, R^*f_1, R^*f_2, \ldots, R^*f_{m_2}. \tag{36}$$

Assuming these vectors to be dependent, then a relation of the following form should hold

$$g = R^*f,$$

where $g \in H \ominus T\mathfrak{D}_T$, $f \in H \ominus F$ and $f \neq 0$. But then for any $\varphi \in \mathfrak{D}_T$ we shall have

$$(\varphi, f) = (T\varphi, R^*f) = (T\varphi, g) = 0.$$

Thus as a result of the density of $\mathfrak{D}_T$ in H, we conclude that $f = 0$. Inequality (35) is thus proved.

Let us now show that

$$\operatorname{Def} T(\mathfrak{D}_T \cap F) \leqslant m_1 + m_2.$$

To prove this inequality it is sufficient to show that if a vector f is orthogonal to all the vectors (36), then it belongs to the subspace $T(\mathfrak{D}_T \cap F)$. To this purpose, let

$$(f, g_j) = 0 \qquad (j = 1, 2, \ldots, m_1)$$

and

$$(f, R^*f_k) = 0 \qquad (k = 1, 2, \ldots, m_2).$$

From the first m_1 equalities it follows that $f \in T\mathfrak{D}_T$, i.e., $f = T\varphi$ $(\varphi \in \mathfrak{D}_T)$, and from the latter m_2 equalities it follows that $T^{-1}f \in F$. A comparison of these two conclusions shows that $f \in T(\mathfrak{D}_T \cap F)$, thus proving the lemma.

Theorem 21 /31 (1)/. *Let T_1 and T_2 be closed linear operators with domains of definition dense in H. Furthermore let these operators possess the regular-type point $\lambda = 0$. Let* $\operatorname{Def} T_1\mathfrak{D}_{T_1} < \infty$ *and* $\operatorname{Def} T_2\mathfrak{D}_{T_2} < \infty$.

Under these conditions the operator T_1T_2 is closed and has a domain of definition dense in H. This operator possesses the regular-type point $\lambda = 0$ and

$$\operatorname{Def} T_1T_2\mathfrak{D}_{T_1T_2} = \operatorname{Def} T_1\mathfrak{D}_{T_1} + \operatorname{Def} T_2\mathfrak{D}_{T_2}. \tag{37}$$

Proof. Let us show the closure of the product T_1T_2 and the density of the manifold $\mathfrak{D}_{T_1T_2}$ in H. For this purpose we represent $\mathfrak{D}_{T_1T_2}$ in the form

$$\mathfrak{D}_{T_1T_2} = T_2^{-1}(\mathfrak{D}_{T_1} \cap T_2\mathfrak{D}_{T_2}). \tag{38}$$

Since the subspace $F = T_2\mathfrak{D}_{T_2}$ has a finite defect number, then according to Lemma 3, the intersection $\mathfrak{D}_{T_1} \cap T_2\mathfrak{D}_{T_2}$ is dense in the subspace $F = T_2\mathfrak{D}_{T_2}$. When applying the bounded operator T_2^{-1} we obtain a manifold which is dense in $\mathfrak{D}_{T_2}$, and, consequently, dense in H.

To prove the closure of the operator T_1T_2, we apply this operator to both sides of equality (38), obtaining

$$T_1T_2\mathfrak{D}_{T_1T_2} = T_1(\mathfrak{D}_{T_1} \cap T_2\mathfrak{D}_{T_2}).$$

According to Lemma 4 the manifold

$$T_1(\mathfrak{D}_{T_1} \cap F) = T_1(\mathfrak{D}_{T_1} \cap T_2\mathfrak{D}_{T_2})$$

is closed, thus proving the closure of the operator T_1T_2.

It is obvious that the zero belongs to the regularity field of the operator T_1T_2. Finally, formula (37) follows directly from Lemma 4.

From the above theorem the following theorem may be proved.

Theorem 22. *If a A is a symmetric operator with a finite defect number (m, n), then A^2 is also a symmetric operator, and its defect number is $(m+n, m+n)$.*

Proof. From Theorem 21 it follows that the operator

$$A^2 + I = (A - iI)(A + iI)$$

defined on a manifold dense in H is closed. It also follows that

$$\operatorname{Def}(A^2 + I)\mathfrak{D}_{A^2} = \operatorname{Def}(A - iI)\mathfrak{D}_A + \operatorname{Def}(A + iI)\mathfrak{D}_A = m + n.$$

We obtain therefore that the operator A^2 is closed and defined on a manifold dense in H. Since in addition, the operator A^2 is symmetric and the point $\lambda = -1$ belongs to its regularity field, then

Q. E. D.

$$\operatorname{Def} A^2 = \operatorname{Def}(A^2 + I)\mathfrak{D}_{A^2} = m + n.$$

This theorem may be easily generalized to any arbitrary natural power of a symmetric operator with a finite defect number. In particular, for any symmetric operator with equal defect numbers we obtain the formula

$$\operatorname{Def} A^n = n \operatorname{Def} A.$$

Theorem 22 proves the assumptions made in the beginning of this subsection regarding the operator A^2. On the problems touched upon in this subsection, see also /72 (1)/.

§ 2. One-dimensional self-adjoint differential operators

7. Decomposition of operators and properties of the spectrum independent of the behavior of the coefficients at finite distances. In all that follows the term $C_k(\alpha, \beta)$-class refers to the set of all complex-valued k-fold continuously differentiable functions of the variable x, defined in an open, finite or infinite interval (α, β). The set of all functions of $C_k(\alpha, \beta)$ which are identically zero near the ends α and β of the interval will be denoted by $C_k(\alpha, \beta)$. The term finite function is usually applied to a function which is identically zero outside some finite interval. The minimum segment inside which a given finite function identically vanishes will be called its support. A finite function which is identically zero for $x > N$ will be called N-finite.

According to our notation, the $C_{2n}(0, \infty)$-class is the set of all complex-valued $2n$-fold continuously differentiable finite functions equal to zero near the point $x = 0$. By $C_k[0, \infty)$ we denote that part of the $C_k(0, \infty)$-class which consists of all the functions which are continuous on the right at the point $x = 0$ together with their derivatives up to the k-th order.

We now introduce the following differential operation on the linear manifold $C_{2n}(0, \infty)$

$$l = \sum_{k=0}^{n} (-1)^{n-k} \frac{d^{n-k}}{dx^{n-k}} p_k(x) \frac{d^{n-k}}{dx^{n-k}}. \tag{1}$$

The coefficients $p_k(x)$ $(k = 0, 1, \ldots, n)$ are assumed here to be real $C_{n-k}[0, \infty)$-class functions. Also $p_0(x) > 0$ for $x \geqslant 0$.

Under more general assumptions regarding the coefficients $p_k(x)$, e.g., when they are measurable and obey the conditions

$$\int_0^x \frac{dt}{|p_0(t)|} < \infty, \quad \int_0^x |p_k(t)|\, dt < \infty, \qquad (k = 1, 2, \ldots, n),$$

the operation l is called the quasi-differential. In this case the role of the $C_k(0, \infty)$-class is played by the manifold of functions which is constructed in a similar way by replacing all derivatives with quasi-derivatives (see /7/). Since all assumptions and proofs are essentially preserved when passing from the differential to the quasi-differential operation, only the former will be considered in the following text. Provided the formulation is properly modified it is possible in the propositions of the present chapter to replace the semi axis $x \geqslant 0$ having one singular end, used here as a standard interval, by any finite or infinite interval with one or two singular ends.

The differential operation (1) defines on $C_{2n}(0, \infty)$ an operator $\dot{L}$ symmetrical in $\mathscr{L}_2(0, \infty)$. $\dot{L}$ is called the minimal unclosed differential operator, defined by the operation (1). Its closure L will be called the minimal differential operator defined by the operation (1). As is known (see /7/), the defect number of this operator satisfies the inequality

$$n \leqslant \operatorname{Def} L \leqslant 2n.$$

Any self-adjoint extension $\tilde{L}$ of the operator L will be called a self-adjoint operator produced by the operation l.

By virtue of Theorem 4, all the self-adjoint operators, generated by a given operation (1) have the same continuous spectrum $C(\tilde{L}) = C(L)$. Moreover, by virtue of Theorem 11, the type of each point of $C(\tilde{L})$ does not depend on the choice of the extension $\tilde{L}$ of the operator L. In this connection we exclude the case of an eigenvalue of infinite multiplicity.

It can be easily seen that the spectrum $S(\tilde{L})$ is never bounded from above. To prove this it is sufficient to calculate the scalar product $(L\varphi_m, \varphi_m)$ for the test system of functions $\varphi_m(x) = \varphi(mx)$, $m = 1, 2, \ldots$, where $\varphi(x)$ is an arbitrary function of $C_{2n}(0, \infty)$, not identically zero, but is equal to zero outside some interval (α, β).

In fact, for $m \to \infty$ we have

$$(\tilde{L}\varphi_m, \varphi_m) = \int_0^\infty \sum_{k=0}^{n} p_k(x) \left|\varphi^{(n-k)}(mx)\right|^2 m^{2(n-k)}\, dx =$$

$$= m^{2n-1} \int_\alpha^\beta p_0\left(\frac{t}{m}\right) \left|\varphi^{(n)}(t)\right|^2 dt + 0\left(m^{2n-1}\right),$$

therefore

$$\lim_{m \to \infty} (\tilde{L}\varphi_m, \varphi_m) = \infty$$

which proves our statement.

Let L_0 denote the operator defined by the equality

$$L_0\varphi = L\varphi$$

for all functions $\varphi \in \mathfrak{D}_L$ which at a given point $\gamma > 0$ satisfy the condition

$$\varphi(\gamma) = \varphi'(\gamma) = \ldots = \varphi^{(2n-1)}(\gamma) = 0. \tag{2}$$

The operator L_0 is obviously an orthogonal sum of the minimal differential operators L_1 and L_γ, defined by the operation (1) in $\mathscr{L}_2(0, \gamma)$ and $\mathscr{L}_2(\gamma, \infty)$ respectively. Condition (2) thus defines a finite decomposition of the operator $\tilde{L}$ according to the formulas

$$\tilde{L} = \tilde{L}_0, \quad L_0 = L_1 \oplus L_\gamma. \tag{3}$$

The point γ is called the decomposition point of the operator L.

Since L_1 in (3) is a regular differential operator, then $C(L_1) = \emptyset$, so that

$$C(L_0) = C(L_1) \cup C(L_\gamma) = C(L_\gamma)$$

and, by the decomposition principle,

$$C(\tilde{L}) = C(L_\gamma). \tag{4}$$

It follows from equality (4) that the continuous part of the spectrum of the operator $\tilde{L}$ does not depend on the behavior of the coefficients of the differential operation (1) at a finite distance.

Extending L_1 and L_γ in the spaces $\mathscr{L}_2(0, \gamma)$ and $\mathscr{L}_2(\gamma, \infty)$ to the self-adjoint operators $\tilde{L}_1$ and $\tilde{L}_\gamma$, we construct the self-adjoint operator $\tilde{L}_0$ by the formula.

$$\tilde{L}_0 = \tilde{L}_1 \oplus \tilde{L}_\gamma.$$

The given operator $\tilde{L}$ and the operator $\tilde{L}_0$ are finite dimensional extensions of the same operator L_0. Therefore, according to Theorem 11, the types of each point of the continuous part of the spectrum of the operators $\tilde{L}$ and $\tilde{L}_0$ coincide. Furthermore, since in each finite interval the operator $\tilde{L}_1$ has only a finite number of eigenvalues, the type of each point of the continuous spectrum of the operator $\tilde{L}_0$ coincides with the type of the same point in the continuous spectrum of the operator $\tilde{L}_\gamma$.

Thus the type of each point of $C(\tilde{L})$ coincides with the type of the same point in $C(\tilde{L}_\gamma)$, and, consequently, the type of each point of the continuous part of the spectrum of the operator $\tilde{L}$ does not depend on the behavior of the coefficients of operation (1) at a finite distance.

Similarly, making use of Theorem 14, it can be shown that the boundedness or unboundedness from below of the spectrum $S(\tilde{L})$ is also independent of the behavior of the coefficients at a finite distance.

8. Small and relatively small perturbations of differential operations. Let us begin with an examination of the two-term differential operation

$$l[y] = (-1)^n \frac{d^{2n}y}{dx^{2n}} + q(x)\, y, \tag{5}$$

which after the introduction of the perturbation $\eta(x)\, y$ takes the form

$$l_\eta[y] = (-1)^n \frac{d^{2n}y}{dx^{2n}} + q(x)\, y + \eta(x)\, y. \tag{6}$$

The following theorem establishes the invariance of the continuous part of the spectrum of the operator $\tilde{L}$ with respect to small perturbations.

Theorem 23 /31 (3)/. If $\tilde{L}$ and $\tilde{L}_\eta$ are operators generated by the operations (5) and (6) respectively, and if

$$\lim_{x \to \infty} \eta(x) = 0, \tag{7}$$

then

$$C(\tilde{L}_\eta) = C(\tilde{L}).$$

Proof. Decomposing the operators $\tilde{L}_\eta$ and $\tilde{L}$ at some point γ, we obtain

$$\tilde{L} = \tilde{L}_0, \qquad L_0 = L_1 \oplus L_\gamma$$

and

$$\tilde{L}_\eta = \tilde{L}_{\eta 0}, \quad L_{\eta 0} = L_{\eta 1} \oplus L_{\eta\gamma},$$

so that

$$C(\tilde{L}) = C(L_\gamma) \quad \text{and} \quad C(\tilde{L}_\eta) = C(L_{\eta\gamma}).$$

Choosing the decomposition point γ so that for $x > \gamma$ we have $|\eta(x)| < \delta$, and using Theorem 7, we find that the δ-neighborbood of each point $\lambda \in C(\tilde{L})$ contains points of $C(\tilde{L}_\eta)$ and, conversely, the δ-neighborhood of each point $\lambda \in C(\tilde{L}_\eta)$ contains points of $C(\tilde{L})$. Since the sets $C(\tilde{L})$ and $C(\tilde{L}_\eta)$ do not depend on the number $\delta > 0$, which may be taken arbitrarily small, then $C(\tilde{L}_\eta) = C(\tilde{L})$, Q. E. D.

It would be incorrect to say that the types of the points of $C(\tilde{L})$ are the same as the types of the points of $C(\tilde{L}_\eta)$, but if we require, in addition to (7), the fulfillment of the inequality $\eta(x) \geqslant 0$ for sufficiently large x, then from Theorem 7[bis] the conservation of the kind of the points of $C(\tilde{L})$ of type (1) (No. 3, page 14) easily follows.

The question of whether the above theorem is also correct for the second order differential operation was posed for the first time in /103 (1)/. There instead of condition (7) only the coincidence of the extreme left points of the sets $C(\tilde{L})$ and $C(\tilde{L}_\eta)$ is established.

Examples illustrating the application of Theorem 23 may be found further in Nos. 33 and 34.

Passing now from small perturbations to relatively small perturbations, we shall assume the function $q(x)$ to be semi-bounded from below.

The following theorem generalizes Theorem 23 to the case of relatively small perturbations.

Theorem 24 /13 (5)/. If $\tilde{L}$ and $\tilde{L}_\eta$ are operators generated by the operations (5) and (6) respectively, where $q(x) \geqslant \alpha > 0$ for $x \geqslant 0$, and if

$$\lim_{x \to \infty} \frac{\eta(x)}{q(x)} = 0,$$

then

$$C(\tilde{L}_\eta) = C(\tilde{L}).$$

This theorem is a particular case of the following theorem of M. Sh. Birman, referring to the general operation (1).

Theorem 25 /14 (5)/. Let in (1) $p_0(x) \equiv 1$, $p_k(x) \geqslant 0$ $(k = 1, 2, \ldots, n-1)$, $p_n(x) \geqslant \alpha > 0$ and let the perturbations $\eta_k(x)$ be introduced in the coefficients $p_k(x)$ $(k = 1, 2, \ldots, n)$, such that*

$$\lim_{x \to \infty} \frac{\eta_k(x)}{p_k(x)} = 0 \qquad (k = 1, 2, \ldots, n). \tag{8}$$

If l_η is an operation obtained from (1) after the introduction of these perturbations, then

$$C(\tilde{L}_\eta) = C(\tilde{L}).$$

Proof. Since we intend to use the theorem on relatively complete continuity, the operator generated by operation (1) on the manifold $\mathfrak{D}_L$ of the finite functions of $\mathfrak{D}_L$ will be denoted by $A = \dot{L}$. By K we denote the operator defined on this same manifold by the operation

$$\sum_{k=1}^{n} \frac{d^{n-k}}{dx^{n-k}} \eta_k \frac{d^{n-k}}{dx^{n-k}}.$$

By virtue of Theorem 19, it is sufficient to show the complete continuity of the operator K relative to the positive definite operator A.

* In the quoted paper of M. Sh. Birman /13 (5)/, instead of $p_0(x) \equiv 1$, it was assumed only that $\alpha \leqslant p_0(x) \leqslant \beta$. A perturbation was introduced into $p_0(x)$ as well.

Furthermore for $f \in \mathfrak{D}_L$ we have

$$|(Kf, f)| \leqslant \int_0^\infty \sum_{k=1}^n |\eta_k(x)| \cdot |f^{(n-k)}(x)|^2 dx,$$

then, by virtue of Theorem 17, it is sufficient to show the complete continuity relative to A of the positive operator K_1 generated in $\mathfrak{D}_L$ by the quasi-differential operation

$$\sum_{k=1}^n \frac{d^{n-k}}{dx^{n-k}} |\eta_k| \frac{d^{n-k}}{dx^{n-k}}.$$

This quasi-differential operation is, by virtue of Theorem 16, equivalent to the compactness in the K_1-metric of any A-bounded set of $\mathfrak{D}_L$.

In order to show the compactness of a given A-bounded set F of functions of $\mathfrak{D}_L$, it is sufficient that a finite ε-network may be constructed in the K_1-metric for any $\varepsilon > 0$, for the set F.

Suppose the inequality

$$(Af, f) < M^2 \tag{9}$$

holds for any $f \in F$. In order to construct the above-mentioned ε-network we choose for a given δ, a number γ such that for $x > \gamma$ we have

$$\frac{|\eta_k(x)|}{p_k(x)} < \delta^2 \qquad (k = 1, 2, \ldots, n).$$

This is possible because of equation (8).

Setting

$$(K_1 f, f)_\gamma = \sum_{k=1}^n \int_0^\gamma |\eta_k| \cdot |f^{(n-k)}(x)|^2 dx,$$

we have

$$\begin{aligned}(K_1 f, f) &= (K_1 f, f)_\gamma + \sum_{k=1}^n \int_\gamma^\infty |\eta_k(x)| \cdot |f^{(n-k)}(x)|^2 dx < \\ &< (K_1 f, f)_\gamma + \delta^2 \sum_{k=1}^n \int_\gamma^\infty p_k(x) |f^{(n-k)}(x)|^2 dx \leqslant \\ &\leqslant (K_1 f, f)_\gamma + \delta^2 (Af, f) < (K_1 f, f)_\gamma + M^2 \delta^2. \end{aligned} \tag{10}$$

The following inequality which follows from (9) holds for all $f \in F$

$$\int_0^\gamma |f^{(n)}(x)|^2 dx < M. \tag{11}$$

From inequality (11) and the conditions

$$f^{(k)}(0) = 0 \qquad (k = 0, 1, \ldots, 2n-1) \tag{12}$$

we obtain the inequalities

$$|f^{(k)}(x)| < N \qquad (k = 0, 1, \ldots, 2n-1), \tag{13}$$

where the number N does not depend on x and f.

The compactness of the family of functions F, in the sense of the uniform convergence in the interval $(0, \gamma)$ of the functions, together with their derivatives up to the $(n-2)$ th order as well as the convergence in the mean square of the $(n-1)$th order derivative, follows from relations (12) and (13).

A fortiori the family of functions F is compact in the metric of the quadratic functional

$$(K_1 f, f)_\gamma. \tag{14}$$

There exists therefore in F a finite set of functions R, such that the distance of any function $f \in F$ from this set, in the metric of (14), is smaller than a given number $\delta > 0$.

From this and from inequality (10) it follows that the K_1-distance of any function $f \in F$ from the set R will be smaller than $\delta\sqrt{1+2M^2}$. Choosing

$$\delta < \frac{\varepsilon}{\sqrt{1+2M^2}},$$

we obtain the required ε-network, thus proving the theorem.

9. Test systems of finite functions and the continuous part of the spectrum of a singular operator as the set of accumulation points of the spectra of regular operators. From the results of No. 2 it follows that the character of the spectrum of the operator $\tilde{L}$ is due to the existence of infinite dimensional manifolds, for which the inequality

$$\|(\tilde{L} - \lambda I)\varphi\| < \delta\|\varphi\|, \tag{15}$$

is satisfied, since

$$(\tilde{L}\varphi - \lambda\varphi, \varphi) < 0. \tag{16}$$

It is obvious that the fulfillment of inequalities (15) and (16) in the case of two functions φ and ψ of $\mathfrak{D}_L$, satisfying the conditions

$$(L\varphi, \psi) = 0, \quad (L\varphi, L\psi) = 0,$$

implies the fulfillment of these inequalities in the case of linear combinations of these functions. Correspondingly, in order to obtain a test manifold G, it is sufficient to construct a test system of mutually orthogonal functions $\{\varphi_k\}$ of $\mathfrak{D}_L$ satisfying the conditions

$$(L\varphi_j, \varphi_k) = 0, \quad (L\varphi_j, L\varphi_k) = 0 \qquad (j \neq k;\ j, k = 1, 2, \ldots).$$

As a system of test functions for differential operators it is natural to use sequences of finite functions with disjoint supports.

Such a system always exists, providing there exists an infinite dimensional manifold G for which (15) or (16) is fulfilled.

As an example let us show this in the case of inequality (15). Since the operator $\tilde{L}$ is a finite dimensional extension of the operator L, then the existence of the infinite dimensional manifold $G \subset \mathfrak{D}_L$ for which inequality (15) holds follows from the existence of the infinite dimensional manifold $\tilde{G}$ on which inequality (15) is also fulfilled.

Let $f_1 \in G$. Since the operator L is constructed to be the closure of the operator L, it is possible to find a finite function $\varphi_1 \in C_{2n}(0, \infty)$ which is so close to f_1 that inequality (15) will hold for φ_1 also.

Let us now choose the point γ_1 to the right of the support of the finite function φ_1 and decompose the operator $\tilde{L}$ at the point $x = \gamma_1$, so that

$$\tilde{L} = \tilde{L}_0, \quad L_0 = L_1 \oplus L_{\gamma_1}.$$

By extending the operators L_1 and L_{γ_1} up to the self-adjoint operators $\tilde{L}_1$ and $\tilde{L}_{\gamma_1}$ in $\mathscr{L}_2(0, \gamma_1)$ and $\mathscr{L}_2(\gamma_1, \infty)$, we construct the operator

$$L_0' = \tilde{L}_1 \oplus \tilde{L}_{\gamma_1}.$$

If follows from inequality (15) that in the interval $(\lambda_0-\delta,\ \lambda_0+\delta)$ there is an infinite set of points of the spectrum $S(\tilde{L})$, and consequently, an infinite set of points of the spectrum $S(L_0')$. But we have

$$S(L_0')=S(\tilde{L}_1)\cup S(\tilde{L}_{\gamma_1}),$$

and in each finite interval there is only a finite number of points of $S(\tilde{L}_1)$, so that in the interval $(\lambda_0-\delta,\ \lambda_0+\delta)$ there should be an infinite set of points of the spectrum of the operator $\tilde{L}_{\gamma_1}$. Hence, as previously shown for the operator $\tilde{L}$, it follows that there exists a finite function $\varphi_2\in C_{2n}(\gamma_1,\ \infty)$ for which inequality (15) holds. The supports of the finite functions φ_1 and φ_2 do not intersect.

Continuing this process indefinitely, we obtain the required sequence of test functions.

Let $N(t)$ be a function defined for all positive values of t. If a number $d\geqslant 0$ exists such that for any t' and t'' with $t''>t'$, the inequality

$$N(t'')\geqslant N(t')-d$$

is satisfied, we say that $N(t)$ is monotonic increasing in the main. If $d=0$, the definition of monotonic increasing in the main reduces to the ordinary definition of a monotonic non-increasing function.

Let a regular self-adjoint operator $\tilde{L}_t$ be defined in the interval $(0,\ t)$, $t>0$, by means of operation (1) and certain boundary conditions. We denote its spectrum by S_t. Furthermore, let $N(t;\ \lambda_0,\ \delta)$ be the number of points of the spectrum S_t lying in the interval $(\lambda_0-\delta,\ \lambda_0+\delta)$.

We shall show that the function $N(t;\ \lambda_0,\ \delta)$ is monotonic increasing in the main.

Suppose that for some $t=t'$ we have $N(t';\ \lambda_0,\ \delta)=m>2n$, then there must exist an m-dimensional subspace $\tilde{G}\subset\mathfrak{D}_{\tilde{L}_t}$ in which

$$\|(\tilde{L}_t-\lambda_0 I)f\|<\delta\|f\|, \tag{17}$$

and since

$$\dim\mathfrak{D}_{\tilde{L}_{t'}}=2n\ (\mathrm{mod}\ \mathfrak{D}_{L_{t'}}),$$

there exists an $m-2n$-dimensional subspace $G_{t'}\subset\mathfrak{D}_{L_{t'}}$ in which inequality (17) also holds. If now $t''>t'$, then obviously $G_{t'}\subset\mathfrak{D}_{\tilde{L}_{t''}}$, and from (17) by virtue of Theorem 9^{bis} we obtain

$$N(t'';\ \lambda_0,\ \delta)\geqslant m-2n,$$

which is what we wanted to show.

In the following theorem it is shown that the continuous part, $C(\tilde{L})$, of the spectrum of a self-adjoint operator $\tilde{L}$, defined on the semi-axis $x\geqslant 0$ by operation (1), may be considered as the set of the accumulation points of the spectra S_t of the operators L_t for $t\to\infty$.

Theorem 26. A necessary and sufficient condition for the point λ_0 to belong to $C(\tilde{L})$ is that the limit relation

$$\lim_{t\to\infty}N(t;\ \lambda_0,\ \delta)=\infty \tag{18}$$

is satisfied for any $\delta>0$.,

Proof. If (18) holds, then by the same reasoning as in the proof of the monotonic increase in the main of the function $N(t;\ \lambda_0,\ \delta)$, it is possible to find in $\mathfrak{D}_{\tilde{L}}$ manifolds of as large a dimension as we wish, for which we have

$$\|(\tilde{L}-\lambda_0 I)\varphi\|<\delta\|\varphi\|. \tag{19}$$

and, therefore, it follows that the set of points of the spectrum $S(\tilde{L})$ lying in the interval $(\lambda_0-\delta, \lambda_0+\delta)$ is infinite. Since the number $\delta>0$ is arbitrary it follows that $\lambda_0\in C(\tilde{L})$. Conversely, if $\lambda_0\in C(\tilde{L})$, then for any $\delta>0$ there exists an infinite dimensional subspace $\tilde{G}\subset\mathfrak{D}_\chi$, for which (19) holds, and since

$$\dim \mathfrak{D}_\chi \leqslant 2n \qquad (\mathrm{mod}\, \mathfrak{D}_L),$$

inequality (19) is satisfied also on some infinite dimensional manifold $G\subset\mathfrak{D}_L$.

Let us choose in G a linearly independent system of functions $\{f_k\}_{k=1}^m$; m is any natural number. Since the operator L is the closure of the operator $\dot{L}$, defined on $\dot{C}_{2n}(0, \infty)$, it is possible to replace each of the m functions f_k $(k=1, 2, \ldots, m)$, by a finite function $\varphi_k\in\mathfrak{D}_L$, so close that the system of functions $\{\varphi_k\}_{k=1}^m$ is linearly independent and inequality (19) is satisfied on its linear envelope.

Since for sufficiently large t we have $\varphi_k\in\mathfrak{D}_{L_t}$ $(k=1, 2, \ldots, m)$, it then follows from (19) that the operator $\tilde{L}_t$ has in the interval $(\lambda_0-\delta, \lambda_0+\delta)$ no less than m eigenvalues. Since the natural number m is arbitrary, the theorem is proved.

Let us denote by $N(t; \mu)$ the number of eigenvalues of the operator $\tilde{L}_t$, which are smaller than μ. Obviously,

$$N(t; \lambda_0, \delta)=N(t; \lambda_0+\delta)-N(t, \lambda_0-\delta).$$

Furthermore in analogy with the result for $N(t, \lambda_0-\delta)$, it can be shown that the function $N(t; \mu)$ is monotonic increasing in the main and that the following theorem holds.

Theorem 27. A necessary and sufficient condition for the set of points of the spectrum of the operator $\tilde{L}$ situated to the left of the point μ to be infinite is that the following limit relation be satisfied

$$\lim_{t\to\infty} N(t, \mu)=\infty.$$

In order for the spectrum $S(\tilde{L})$ to be unbounded from below it is necessary and sufficient that this limit relation be satisfied for any μ.

10. The quadratic functional, and general criteria for the discreteness of the negative part of the spectrum. Operation (1) gives rise to the quadratic functional

$$\Phi[y]=\int_0^\infty \sum_{k=0}^n p_k(x)\,|y^{(n-k)}(x)|^2\,dx, \tag{20}$$

which in what follows is applied only to finite functions.

For $\varphi\in\mathfrak{D}_L$ we have

$$\Phi[\varphi]=(L\varphi, \varphi);$$

equality (20) however defines the functional $\Phi[y]$ on a wider manifold to be described later on.

From Theorem 13 we obtain the following theorem which relates the character of the negative part of the spectrum of the operator $\tilde{L}$ to the set of values of the functional (20).

Theorem 28 /31 (9)/. A necessary and sufficient condition for the set $(-\infty, 0)\cap S(\tilde{L})$ not to have non-vanishing limit points is that for any $\varepsilon>0$ there exists a value $N=N_\varepsilon$ for which the quadratic functions

$$\Phi_\varepsilon[y]=\int_N^\infty \sum_{k=0}^n p_k(x)\,|y^{(n-k)}(x)|^2\,dx+\varepsilon\int_N^\infty |y(x)|^2\,dx$$

are non-negative on all the N-finite functions of $\mathfrak{D}_L$.

A necessary and sufficient condition for the set $(-\infty, 0) \cap S(\tilde{L})$ *to be finite is that there exists an* N *such that the functional* $\Phi[y]$ *is non-negative on all the* N*-finite functions of* $\mathfrak{D}_L$.

Proof. If the set $(-\infty, 0) \cap S(\tilde{L})$ has no limit point other than zero, then for any $\varepsilon > 0$ the set $(-\infty, -\varepsilon) \cap S(\tilde{L})$ is finite. Suppose that for some $\varepsilon_0 > 0$ there does not exist a number N having the properties indicated in the statement of the theorem; then there would exist in $\mathfrak{D}_L$ an infinite sequence of finite functions with disjoint supports on which the functional Φ_ε would be negative. Hence, by virtue of Theorem 13, it follows that the set $(-\infty, -\varepsilon_0) \cap S(\tilde{L})$ must be infinite. That is, there exists in the set $(-\infty, 0) \cap S(\tilde{L})$ a limit point different from zero.

Conversely, if for any $\varepsilon > 0$ there exists $N = N_\varepsilon$ with the property indicated in the statement of the theorem, then the set $(-\infty, \varepsilon) \cap S(\tilde{L})$ is finite for any $\varepsilon > 0$. In fact, assuming that for some $\varepsilon_0 > 0$ this set is infinite, we obtain that for any N the set $(-\infty, -\varepsilon_0) \cap S(\tilde{L}_N)$ is infinite. But then, by virtue of Theorem 13, the functional Φ_{ε_0} would not be non-negative.

The second part of the theorem can be proved on taking $\varepsilon_0 = 0$ in the above analysis.

Remark. From Theorem 28 it follows that in order for the set $(-\infty, \mu) \cap S(\tilde{L})$ to be finite, it is necessary and sufficient that the functional $\Phi[y] - \mu(y, y)$ be non-negative for some N on all the N-finite functions of $\mathfrak{D}_L$. In particular, a necessary and sufficient condition for the boundedness from below and discreteness of the whole spectrum $S(\tilde{L})$ is that there exists such a number $N = N_\mu$ for any arbitrarily large μ.

It also follows from Theorem 28 that if for a given coefficient $q(x)$ the spectrum of the operator generated by the operation (5) is bounded from below and discrete for $n = 1$, then so is the spectrum of the operator generated by this operation for any n. The proof follows from the equation

$$(-1)^n y^{2n} + qy = \{(-1)^n y^{(2n)} + y''\} + \{-y'' + q(x)\, y\}.$$

The first term on the right-hand side of the above equation defines the semi-bounded minimal operator (its lower bound is $\alpha = -1$).

Returning to the problem of the domain of definition $\mathfrak{D}_\Phi$ of the functional $\Phi[y]$, we confine ourselves solely to finite functions equal to zero in the neighborhood of the point $x = 0$. It is seen from formula (20) that the minimum smoothness requirements for any function y of $\mathfrak{D}_\Phi$ are that it belongs to the $C_{n-1}(0, \infty)$-class, and the existence of its n-th derivative belonging to $\mathscr{L}_2(0, \infty)$. The following lemma shows that the extension of the functional $\Phi[y]$ from the manifold $C_{2n}(0, \infty)$ to the manifold of the functions satisfying the above weak smoothness requirements is unessential. This means that these functions are contained in the domain of definition of the unbounded functional $\Phi[y]$, when it is extended by continuity from the initial manifold $\mathring{C}_{2n}(0, \infty)$.

Lemma 5 /53 (3)/. If $z(x)$ *is a finite function with the support* $[\alpha, \beta]$, *belonging to the* $C_{n-1}[\alpha, \beta]$*-class, having a derivative* $z^{(n)}(x) \in \mathscr{L}_2(\alpha, \beta)$ *and satisfying the* $2n$ *boundary conditions*

$$z^{(k)}(\alpha) = z^{(k)}(\beta) = 0 \qquad (k = 0, 1, 2, \ldots, n-1), \tag{21}$$

then for any $\varepsilon > 0$ *there exists a finite function* $y(x)$ *with the same support, belonging to the* $C_{2n}(\alpha, \beta)$*-class and satisfying the* $4n$ *boundary conditions*

$$y^{(k)}(\alpha) = y^{(k)}(\beta) = 0 \qquad (k = 0, 1, \ldots, 2n-1), \tag{22}$$

such that

$$|\Phi[z]-\Phi[y]|<\varepsilon. \tag{23}$$

Proof. Let us consider together with the functional $\Phi[y]$ the simplest quadratic functional of the kind

$$F[y]=\int_{\alpha}^{\beta}|y^{(n)}(x)|^2\,dx.$$

The fundamental character of the sequence in the (generally indefinite) Φ-metric follows from the fundamental nature of any sequence $\{y_j(x)\}$ of functions of $C_{2n}(\alpha,\beta)$ in the F-metric, since for $0\leqslant r<n$ we have

$$\begin{aligned}&|y_j^{(r)}(x)-y_k^{(r)}(x)|\leqslant\\ &\leqslant\frac{1}{(n-r-1)!}\int_{\alpha}^{x}(x-s)^{n-r-1}|y_j^{(n)}(s)-y_k^{(n)}(s)|\,ds\leqslant\\ &\leqslant\frac{(\beta-\alpha)^{n-r-\frac{1}{2}}}{(n-r-1)!}\left\{\int_{\alpha}^{\beta}|y_j^{(n)}(s)-y_k^{(n)}(s)|^2\,ds\right\}^{\frac{1}{2}}\end{aligned} \tag{24}$$

Therefore to prove the theorem it is permissible to replace the functional Φ by the functional F in formula (23).

It is now sufficient to show that the manifold of all the functions $z(x)$ satisfying the conditions of the lemma is the complete space in the F-metric, and that from the F-orthogonality of one of these functions $z(x)$ to all the functions $y\in C_{2n}(\alpha,\beta)$, satisfying the boundary conditions (22), it follows that $z(x)\equiv 0$.

Passing to the proof of the above mentioned completeness, we assume that the sequence $\{z_j(x)\}$ is F-convergent, so that in the metric of $\mathscr{L}_2(\alpha,\beta)$ have

$$\lim_{j\to\infty}z_j^{(n)}(x)=h(x).$$

Taking

$$z(x)=\frac{1}{(n-1)!}\int_{\alpha}^{x}(x-s)^{n-1}\,h(s)\,ds,$$

so that we have $z(x)\in C_{n-1}(\alpha,\beta)$, $z^{(n)}(x)\in\mathscr{L}_2(\alpha,\beta)$ and $z^{(k)}(\alpha)=0$ $(k=0,1,\ldots,n-1)$. Furthermore by virtue of an inequality of the type (24), we have for $0\leqslant r<n$ that the sequence of functions $\{z_j^{(r)}(x)\}$ is uniformly convergent, i.e.,

$$\lim_{j\to\infty}z_j^{(r)}(x)=z^{(r)}(x),$$

in particular,

$$z^{(r)}(\beta)=\lim_{j\to\infty}z_j^{(r)}(\beta)=0\qquad(r=0,1,\ldots,n-1).$$

Thus, together with the F-convergent sequence of the functions $\{z_j(x)\}$ the limit function $z(x)$ satisfies all the conditions of the lemma and the F-completeness is established.

Let $z(x)$ be a function satisfying all the conditions of the lemma; $z(x)$ is F-orthogonal to the manifold of all the functions $y(x)$ of $C_{2n}(\alpha,\beta)$, and satisfies the boundary conditions (22), so that

$$\int_{\alpha}^{\beta}z^{(n)}(x)\,\overline{y^{(n)}(x)}\,dx=0.$$

Hence, integrating by parts under conditions (21), we obtain

$$\int_{\alpha}^{\beta} z(x)\,\overline{y^{(2n)}(x)}\,dx = 0.$$

From the above relation it follows that the function $z(x)$ belongs to the domain of definition of the operator which is adjoint to the operator defined by $2n$-fold differentiation. The latter operator has a minimum domain of definition in $\mathcal{L}_2(\alpha, \beta)$. The function $z(x)$ is therefore absolutely continuous together with its derivatives up to the $(n-1)$-th order and in the metric $\mathcal{L}_2(\alpha, \beta)$, we have

$$z^{(2n)}(x) = 0. \tag{25}$$

From relation (25) and from the boundary conditions (22) it follows that $z(x) \equiv 0$, and the lemma is completely proved.

In the following subsection we make use of a particular case of this lemma.

11. Localization principles. This term refers to theorems which enable us to predict the spectral properties of a singular operator L, from the spectral properties of the family $\mathcal{L}_\Delta$ of the regular operators generated by the same operation (1) on any fixed interval Δ (in the multi-dimensional case — in any cube of a given side). The first theorem of this kind for one-dimensional and multi-dimensional Schroedinger operators was obtained by A. M. Molchanov /70/. The importance of the localization principles lies in the fact that they reduce the investigation of certain properties of the spectrum $S(L)$ of a singular operator to the study of the corresponding family of regular operators on a fixed finite interval, or in the multi-dimensional case — inside a fixed cube.

We give below two localization principles for the operation

$$l[y] = -y'' + q(x)\,y \tag{26}$$

defined in $\mathcal{L}_2(-\infty, \infty)$. These were recently established by R.S. Ismagilov /46/. The proof of these principles is based on the following construction.

Let $\theta_0(x)$ denote any function of $C_2[-h, h]$ which is equal to zero for $|x| \geqslant h$, and satisfies the relation

$$\theta_0^2(x) + \theta_0^2(x-h) = 1 \qquad (0 \leqslant x \leqslant h). \tag{27}$$

We assume also that

$$\theta_k(x) = \theta_0(x - kh) \qquad (k = \pm 0, 1, 2, \ldots)$$

and

$$u_k(x) = y(x)\,\theta_k^2(x), \quad v_k(x) = y(x)\,\theta_k(x)\,\theta_{k+1}(x).$$

It is obvious that the function $u_k(x)$ is zero outside the interval $(k-1)h < x < (k+1)h$, and that outside the interval $kh < x < (k+1)h$ the function $v_k(x)$ is zero. In the interval $kh < x < (k+1)h$ we have for any finite function $y \in \mathfrak{D}_L$

$$(u_k')^2 = y'^2\theta_k^4 + 4yy'\theta_k^2\theta_k\theta_k' + 4y^2\theta_k^2\theta_k'^2,$$

$$2(v_k')^2 = 2y'^2\theta_k^2\theta_{k+1}^2 + 4yy'\theta_k\theta_{k+1}(\theta_k'\theta_{k+1} + \theta_k\theta_{k+1}') + 2y^2(\theta_k'\theta_{k+1} + \theta_k\theta_{k+1}')^2,$$

$$(u_{k+1}')^2 = y'^2\theta_{k+1}^4 + 4yy'\theta_{k+1}^2\theta_{k+1}\theta_{k+1}' + 4y^2\theta_{k+1}^2\theta_{k+1}'^2,$$

so that

$$u_k'^2 + u_{k+1}'^2 + 2v_k'^2 = y'^2(\theta_k^2 + \theta_{k+1}^2)^2 + \\ + 4yy'\theta_k^2(\theta_k\theta_k' + \theta_{k+1}\theta_{k+1}') + 4yy'\theta_{k+1}^2(\theta_k\theta_k' + \theta_{k+1}\theta_{k+1}') + \\ + 4y^2\theta_k^2\theta_k'^2 + 2y^2(\theta_k'\theta_{k+1} + \theta_k\theta_{k+1}')^2 + 4y^2\theta_{k+1}^2\theta_{k+1}'^2. \quad (28)$$

But since

$$\theta_k^2 + \theta_{k+1}^2 = 1$$

and, consequently,

$$\theta_k\theta_k' + \theta_{k+1}\theta_{k+1}' = 0,$$

equality (28) takes the form

$$u_k'^2 + u_{k+1}'^2 + 2v_k'^2 = y'^2 + 4y^2\theta_k^2\theta_k'^2 + 2y^2(\theta_k'\theta_{k+1} + \theta_k\theta_{k+1}')^2 + \\ + 4y^2\theta_{k+1}^2\theta_{k+1}'^2 = y'^2 - 8y^2\theta_k\theta_k'\theta_{k+1}\theta_{k+1}' + 2y^2(\theta_k'\theta_{k+1} + \theta_k\theta_{k+1}')^2 = \quad (29) \\ = y'^2 + 2y^2(\theta_k'\theta_{k+1} - \theta_k\theta_{k+1}')^2.$$

Further

$$u_k^2 + u_{k+1}^2 + 2v_k^2 = y^2\theta_k^4 + y^2\theta_{k+1}^4 + 2y^2\theta_k^2\theta_{k+1}^2 = \\ = y^2(\theta_k^2 + \theta_{k+1}^2)^2 = y^2. \quad (30)$$

From (29) and (30) we have that for any finite function $y \in \mathfrak{D}_L$ the following relation holds

$$(l[y], y) = \sum_{k=-\infty}^{\infty} (l[u_k], u_k) + 2\sum_{k=-\infty}^{\infty} (l[v_k], v_k) - \\ - 2\sum_{k=-\infty}^{\infty} \int_{kh}^{(k+1)h} y^2(\theta_k'\theta_{k+1} - \theta_k\theta_{k+1}')^2\,dx. \quad (31)$$

The following lemma, based on identity (31), shows that if the smallest eigenvalue of the operator (26), on any section of length $2h$ of a finite interval Ω, is higher than some bound, then the smallest eigenvalue of operation (26) on the interval Ω is also higher than some bound, independent of the length of this interval.

Lemma 6 /46/. Let $\lambda(\Omega)$ be the first eigenvalue of the operation (26) in the interval Ω with zero boundary conditions on the ends of this interval. Let $\lambda(\omega)$ have the same meaning for the interval ω.

For any number $|\omega| > 0$, smaller than the length of the interval Ω, an interval $\omega \subset \Omega$ of length $|\omega|$ can be found, for which we have

$$\lambda(\omega) < \lambda(\Omega) + \frac{32}{|\omega|^2}. \quad (32)$$

Proof follows from identity (31), if we impose as an additional requirement on the function $\theta_0(x)$ that

$$|\theta_0'(x)| < \frac{1}{h}.$$

This requirement is compatible with (27). Set $2h = |\omega|$, and suppose that the support of the finite function $y(x)$ lies inside Ω. Further, make use of the extremal property of the first eigenvalue, taking into account in this connection Lemma 5.

Identity (31) then gives

$$(l[y],\ y) \geqslant \sum_{k=-\infty}^{\infty} (l[u_k],\ u_k) + 2\sum_{k=-\infty}^{\infty} (l[v_k],\ v_k) - \frac{32}{|\omega|^2}(y,\ y), \tag{33}$$

but by virtue of (30) we have

$$(y,\ y) = \sum_{k=-\infty}^{\infty} (u_k,\ u_k) + 2\sum_{k=-\infty}^{\infty} (v_k,\ v_k). \tag{34}$$

Choosing the function $y(x)$ so that for some $\delta > 0$

$$(Ly,\ y) < [\lambda(\Omega) + \delta](y,\ y),$$

we obtain from (33) and (34) the inequality

$$\sum_{k=-\infty}^{\infty} [(l[u_k],\ u_k) - \alpha(u_k,\ u_k)] + 2\sum_{k=-\infty}^{\infty} [(l[v_k],\ v_k) - \alpha(v_k,\ v_k)] < 0, \tag{35}$$

where

$$\alpha = \lambda(\Omega) + \delta + \frac{32}{|\omega|^2}.$$

Since at least one of the terms in the first or second summation on the left-hand side of relation (35) should be negative, and the number $\delta > 0$ may be taken arbitrarily small, the lemma is proved.

From Lemma 6 one can easily deduce the localization principles expressed by Theorems 29 and 30.

Theorem 29 /46/. A necessary and sufficient condition for the semi-boundedness of the operator L generated by operation (26) in $\mathfrak{L}_2(-\infty, \infty)$ is that the set of numbers $\lambda(\omega)$ be bounded from below. ω runs through the totality of all segments of a fixed length.

Proof. If the operator L is bounded from below by the number $\alpha > -\infty$, then the functional $(Ly,\ y) - \alpha(y,\ y)$ will be non-negative on all the functions $y \in \mathfrak{D}_L$, and in particular, on the finite functions with the support ω; hence follows the inequality $\lambda(\omega) > \alpha$.

Conversely, if $\lambda(\omega) > \alpha > -\infty$ for all ω, then for any finite function $y \in \mathfrak{D}_L$ with an arbitrary support Ω, we have by virtue of Theorem 6 that $(Ly,\ y) > \alpha - \frac{32}{|\omega|^2}$, so that the operator L is semi-bounded.

Theorem 30 /46/. A necessary and sufficient condition for the semi-boundedness and discreteness of the spectrum of the operator L is that the value of $\lambda(\omega)$ tends to infinity when the segment ω is displaced to infinity while preserving its length.

Proof. Let the spectrum $S(L)$ be semi-bounded and discrete. Then for any given $\mu > 0$ a decomposition point $N = N_\mu$ can be found, such that on all the N-finite functions of $\mathfrak{D}_L$ we shall have $(L_N y,\ y) > \mu(y,\ y)$. If this were not so, then we could construct an infinite sequence of finite functions with disjoint supports, on which we would have $(Ly,\ y) < \mu(y,\ y)$ leading to $C(L) \cap (-\infty, \mu) \neq \neq \emptyset$. Thus, for all the intervals ω situated to the right of the point $N = N_\mu$ we shall have $\lambda(\omega) > \mu$. This shows that the condition is necessary.

Conversely, suppose that for arbitrarily chosen $\mu > 0$ we have $\lambda(\omega) > \mu$ for all the intervals ω lying to the right of the point $N = N_\mu$. By virtue of Lemma 6 we then have for all the N-finite functions of $\mathfrak{D}_L$ that

$$(L_N y,\ y) > \left(\mu - \frac{32}{|\omega|^2}\right)(y,\ y).$$

It follows from this inequality that

$$C(L_N) \cap \left(-\infty, \mu - \frac{32}{|\omega|^2}\right) = \emptyset$$

and by virtue of the decomposition principle also that

$$C(L) \cap \left(-\infty, \mu - \frac{32}{|\omega|^2}\right) = \emptyset,$$

hence for $\mu \to \infty$ we obtain that $C(L) = \emptyset$. Q. E. D.

12. The negative part of the spectrum and the oscillatory properties of differential operations. For $n = 1$ there exists a special relation between the oscillatory properties of the solutions of the differential equation

$$l[y] = \lambda y \tag{36}$$

and the spectrum of any self-adjoint operator $\tilde{L}$, generated by operation (1). By virtue of this relation the set of points of the spectrum of the operator $\tilde{L}$ preceding the point λ_0 will be finite or infinite depending on whether equation (36) is non-oscillatory or oscillatory for $\lambda = \lambda_0$ (that is, whether each of its solutions has a finite or infinite number of zeros). Passing to any value of n, we note that the arbitrary constants of the general solution of differential equation (36) of $2n$-th order may always be chosen in such a way that this solution has zeros at given $2n - 1$ points. But it is obviously not always possible to find a solution which has zeros at $2n$ given points. In particular it is not always possible to find a solution having n-fold zeros at two given points. The definition of the oscillatory character adopted below is such that in passing from $n = 1$ to any value of n the above relation between the oscillatory character and the properties of the spectrum is preserved.

A differential equation of $2n$-th order (36) will be called oscillatory if for any t, a solution of this equation can be found which has, to the right of the point t, more than one n-fold zero. If this is not so, equation (36) is called non-oscillatory. This term will be applied also to the operation $l[y]$ itself.

For $n = 1$, the definition given above coincides with the usual one. In fact, if the second-order equation (36) is oscillatory in the sense of the above definition, then there exists a sequence of points $\alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \dots$, such that for each interval $[\alpha_k, \beta_k]$ there exists a solution of equation (36) which vanishes at its ends. By virtue of the well-known theorem on the alternation of the zeros, any solution of a second-order differential equation has a root in each of the intervals $[\alpha_k, \beta_k]$, that is, it has an infinity of zeros.

Theorem 31 /31 (9)/. A necessary and sufficient condition for equation of $2n$-th order (36) to be oscillatory for $\lambda = \lambda_0$ is that the part of the spectrum of the operator $\tilde{L}$ lying to the left of the point $\lambda = \lambda_0$ is an infinite set.

Proof. We may assume, without loss of generality, that $\lambda_0 = 0$. If equation (36) is oscillatory for λ_0, then for $x = t_1$ there exists a solution $y_1(x)$ of equation (26) having n-fold zeros at the points α and β, where $\beta > \alpha > t_1$. The function $y_1(x)$ in the interval $[\alpha, \beta]$ may be regarded as the eigenfunction corresponding to the value $\lambda = 0$ of the boundary-value problem, which is determined by the operation l and by the conditions

$$y^{(k)}(\alpha) = y^{(k)}(\beta) = 0 \qquad (k = 0, 1, \dots, n-1).$$

Let $\alpha_1 = \alpha$, and β_1 be any number larger than β.

It follows from variational principles that the problem with the same differential operation l and boundary conditions

$$y^{(k)}(\alpha_1) = y^{(k)}(\beta_1) = 0 \qquad (k = 0, 1, \ldots, n-1)$$

should have an eigenvalue $\lambda_1 < 0$.

Let the corresponding eigenfunction be $f_1(x)$. This function has n-fold zeros at the points α_1 and β_1, and

$$\Phi[f_1] = \lambda_1(f_1, f_1) < 0.$$

Applying Lemma 5 of No. 10, we find a finite function $\varphi_1 \in \mathfrak{D}_L$ with the same support $[\alpha_1, \beta_1]$, for which we also have

$$(L\varphi_1, \varphi_1) = \Phi[\varphi_1] < 0.$$

Choosing now $t_2 > \beta_1$ and iterating the construction, we finally find an infinite sequence of finite functions $\varphi_k \in \mathfrak{D}_L$ with disjoint supports, satisfying the inequality

$$(L\varphi_k, \varphi_k) < 0 \qquad (k = 1, 2, \ldots).$$

Hence, by virtue of Theorem 13, it follows that the negative part of the spectrum of the operator $\tilde{L}$ is an infinite set.

Conversely, if the set $(-\infty, 0) \cap S(\tilde{L})$ is infinite, then for any $\gamma > 0$ the negative part of the spectrum of the operator $\tilde{L}_\gamma$ is also infinite. Hence, by virtue of Theorem 27, it follows that for sufficiently large $\beta > \gamma$, the regular operator $\tilde{L}_{\gamma, \beta}$ generated by operation (1) on the interval $[\gamma, \beta]$ will have a sufficiently large number of negative eigenvalues.

Let t be any positive number. We have to show that for $\lambda = 0$ there exists a solution of equation (36), having to the right of the point t two n-fold zeros.

Let us choose arbitrarily $\alpha_1 > t$ and find a number $\beta > \alpha_1$ such that the operator $\tilde{L}_{\alpha_1, \beta}$ has at least one negative eigenvalue $\lambda(\beta)$. As β decreases, the eigenvalue $\lambda(\beta)$ increases without limit, in a continuous manner. There exists therefore such a value β_1 $(\alpha_1 < \beta_1 < \beta)$, for which $\lambda(\beta_1) = 0$.

Thus the boundary-value problem defined by the operation (1) and by the conditions

$$y^{(k)}(\alpha_1) = y^{(k)}(\beta_1) = 0 \qquad (k = 0, 1, \ldots, n-1).$$

has the eigenvalue $\lambda = 0$. The eigenfunction corresponding to this eigenvalue is the required solution of the differential equation (36) for $\lambda = 0$ since this function has n-fold zeros at the points α_1 and β_1.

It follows from the proof that if the differential operation l is oscillaroty, the first of the n-fold zeros indicated in the definition of oscillatoriness may be chosen arbitrarily.

It also follows from Theorem 31 that if equation (36) is oscillatory for $\lambda = \lambda_0$, it is a fortiori oscillatory for $\lambda > \lambda_0$.

For each operation l one of the following three cases will occur. Either it is non-oscillatory for all λ, or it is oscillatory for all λ, or there exists a value $\lambda = \lambda_0$, such that for $\lambda < \lambda_0$ the operation l is non-oscillatory, and for $\lambda > \lambda_0$ it is oscillatory. In the latter case the operation l may be non-oscillatory or oscillatory for $\lambda = \lambda_0$.

It also follows from Theorem 31 that if operation (1) is non-oscillatory for a given λ, any operation with higher coefficients is a fortiori non-oscillatory for this value of λ.

Finally, from Theorem 31 it follows that the set of all the non-oscillatory operations of a given order with the leading coefficient equal to one is convex. This implies that if the equations

$$(-1)^n y^{(2n)} + \sum_{k=1}^{n} (-1)^{n-k} \frac{d^{n-k}}{dx^{n-k}} p_k^*(x) \frac{d^{n-k}y}{dx^{n-k}} = 0$$

and

$$(-1)^n y^{(2n)} + \sum_{k=1}^{n} (-1)^{n-k} \frac{d^{n-k}}{dx^{n-k}} p_k^{**}(x) \frac{d^{n-k}y}{dx^{n-k}} = 0$$

are non-oscillatory, the equation

$$(-1)^n y^{(2n)} + \sum_{k=1}^{n} (-1)^{n-k} \frac{d^{n-k}}{dx^{n-k}} \times \left[\mu^{**} p_k^*(x) + \mu^* p_k^{**}(x)\right] \frac{d^{n-k}y}{dx^{n-k}} = 0$$

is non-oscillatory for any positive μ^* and μ^{**}, related by the equality $\mu^* + \mu^{**} = 1$. This property was established for the first time by N. Adamov /1/ for $n = 1$ in the case of a periodic function $q(x)$. The proof was accomplished by passing to the corresponding Riccati equation.

In conclusion we give another definition of oscillatory equation, which is equivalent to that used at the beginning of this subsection. This second definition is closer to the ordinary definition of oscillatory second-order differential equations in the sense that it implies the existence of an infinite sequence of zeros of some function $D(x; \lambda)$, constructed with the aid of the solutions of equation (36). For $n = 1$, this function $D(x; \lambda)$ is itself a solution of a differential equation.

To construct the function $D(x; \lambda)$ for a given n, we introduce a n-dimensional linear manifold G of the solutions of equation (36), satisfying for $x = 0$ the conditions

$$\sum_{k=1}^{2n} \alpha_{jk} y^{(k-1)}(0) = 0 \qquad (j = 1, 2, \ldots, n), \tag{37}$$

where the rectangular real matrix,

$$(\alpha_{jk}) \qquad (j = 1, 2, \ldots, n;\ k = 1, 2, \ldots, 2n)$$

is such that conditions (37) together with the conditions

$$y^{(k)}(a) = 0 \qquad (k = 0, 1, \ldots, n-1) \tag{38}$$

determine a self-adjoint extension of the operator, generated by operation (1) in $\mathscr{L}_2(0, a)$. Choosing an arbitrary base in G

$$\varphi_1(x; \lambda),\ \varphi_2(x; \lambda), \ldots, \varphi_n(x; \lambda), \tag{39}$$

we take

$$D(x; \lambda) = \operatorname{Det}\left(\varphi_j^{(k-1)}(x; \lambda)\right)_{j,k=1}^{n} \tag{40}$$

and denote by $r_1, r_2, \ldots, r_m$ the multiplicities of the successive zeros $x_1 < x_2 < \ldots < x_m$ of this Wronskian $D(x; \lambda)$, lying in the interval $(0, a)$. It is obvious that the function $D(x; \lambda)$ is essentially determined by the subspace G itself, since a change of the base (39) would lead only to a multiplication of (40) by a constant non-vanishing factor.

According to a theorem of M. G. Krein /53 (1)/, the number of eigenvalues of the boundary-value problem (37), (38) lying to the left of the point λ satisfies the inequality

$$m \leqslant \nu \leqslant \sum_{j=1}^{m} \sqrt{r_j} \tag{41}$$

where the numbers m, ν, and r_j depend on a, so that $m = m(a)$, $\nu = \nu(a)$ and $r_j = r_j(a)$.

Now if (36) is oscillatory for a given value of λ in the sense of the first definition, then it follows from Theorem 31 that to the left of the point λ there is an infinite set of points of the spectrum of the operator $\tilde{L}$, generated in $\mathscr{L}_2(0, \infty)$ by the operation (1). Hence, on the basis of Theorem 27 we conclude that

$$\lim_{a \to \infty} \nu(a) = \infty, \tag{42}$$

but then, from inequality (41) it follows that

$$\lim_{a \to \infty} \sum_{j=1}^{m(a)} \sqrt{r_j(a)} = \infty,$$

so that

$$\lim_{a \to \infty} m(a) = \infty, \tag{43}$$

that is, the function $D(x; \lambda)$ has an infinite sequence of zeros on the semi-axis $x > 0$.

Conversely, if the function $D(x; \lambda)$ has an infinite sequence of zeros on the semi-axis $x > 0$, that is if (43) holds, then (42) follows from inequality (41). Hence, on the basis of Theorem 28 we conclude that there is an infinity of eigenvalues of the operator $\tilde{L}$ in $\mathscr{L}_2(0, \infty)$ situated to the left of the point λ. Hence, by virtue of Theorem 31, equation (36) is oscillatory.

The above considerations lead to the following alternative. Independently of the choice of the conditions (37), the function $D(x; \lambda)$ for $x > 0$ either has an infinite or only a finite number of zeros.

In accordance with the above account, it is possible to give the following definition of oscillatory equation. This definition is equivalent to the previous one. An equation of $2n$-th order (36) will be called oscillatory, if some (and thus also any other) Wronskian of n-th order (40) has an infinite sequence of positive roots. Conversely, when any Wronskian (40) has only a finite number of positive roots, equation (36) is called non-oscillatory.

In connection with this definition of an oscillatory equation, see also No. 44, Theorem 14.

13. On differential operations on vector-functions. Let a differential operation on vector-functions be defined by

$$l = \sum_{r=0}^{n} (-1)^{n-r} \frac{d^{n-r}}{dx^{n-r}} P_r(x) \frac{d^{n-r}}{dx^{n-r}}, \tag{44}$$

where the coefficients $P_r(x) = \| p_{r; j, k}(x) \|_{j, k=1}^{m}$ $(m < \infty)$ are assumed $(n-r)$-fold continuously differentiable Hermitian matrices, and the matrix $P_0(x)$ is, moreover, assumed positive definite for $x \geqslant 0$.

Let us denote by $\mathscr{L}_2(0, \infty)$ the Hilbert space of the m-component vector-functions $\varphi(x) = \{\varphi_k(x)\}_{k=1}^{m}$ having the scalar product

$$(\varphi, \psi) = \int_0^\infty \sum_{k=1}^{m} \varphi_k(x) \overline{\psi_k(x)}\, dx.$$

The operation l defines an operator $\dot{L}$ (symmetrical in $\mathscr{L}_2(0, \infty)$) on the manifold $C_{2n}(0, \infty)$ of all the $2n$-fold continuously differentiable finite vector-functions equal to zero near the point $x = 0$. The closure of $\dot{L}$ will be denoted by L.

Since any self-adjoint extension $\tilde{L}$ of the operator L generated by operation (44) is finite dimensional, namely $\dim \mathfrak{D}_{\tilde{L}} \leqslant 2mn \pmod{\mathfrak{D}_L}$, all the results of Nos. 7, 9, 10, 12 apply to differential operations on vector-functions of the type (44).

The quadratic functional corresponding to operation (44) has the form

$$\Phi[y]=\int_0^\infty\left\{\sum_{r=0}^{n}\sum_{j,k=1}^{m}p_{r;j,k}(x)\,y_j^{(n-r)}(x)\,\overline{y_k^{(n-r)}(x)}\right\}dx.$$

Operation (44) is called oscillatory, if for any t there exists a solution of the equation

$$l[y]=\lambda y, \tag{45}$$

having, to the right of the point t, more than one n-fold zero. If such a solution does not exist, then operation (44) is called non-oscillatory.

If all the matrices $P_r(x)(r=1,2,\ldots,m)$ are diagonal, equation (45) breaks down into a system of m differential equations of $2n$-th order

$$l_r[y_r]=\lambda y_r \qquad (r=1,2,\ldots,m)$$

with respect to each of the components of the vector-functions $y(x)$ (compare equation (36)). The vector operation (44) will then be oscillatory for $\lambda=\lambda_0$, if at least one of the scalar operations l_r is oscillatory for this value of λ. It will be non-oscillatory for $\lambda=\lambda_0$, if all the operations $l_r(r=1,2,\ldots,m)$ are non-oscillatory for this λ.

V.A. Yakubovich established (see /111 (3)/ that the definition of oscillatory operation considered in this subsection is equivalent to the corresponding definition in the theory of canonic systems of differential equations.

14. A finite-difference analog of the one-dimensional self-adjoint singular differential operator. A discrete analog of the second-order differential operation on the semi-axis $x\geqslant 0$ with a boundary condition of the form $y'(0)=hy(0)$ is the finite-difference operation

$$l[y_k]=\alpha_{k+1}y_{k+1}+\beta_k y_k+\alpha_k y_{k-1} \qquad (k=0,1,2,\ldots), \tag{46}$$

where $y_{-1}=0$.

In the following let the numbers α_k be positive, and the numbers β_k be arbitrary and real.

By introducing the vectors

$$y=\{y_k\}_{k=0}^{\infty} \quad \text{и} \quad z=\{\alpha_{k+1}y_{k+1}+\beta_k y_k+\alpha_k y_{k-1}\}_{k=0}^{\infty},$$

operation (46) can be represented in matrix form with the aid of the equalities

$$z=l[y],$$

$$l=\begin{pmatrix}\beta_0 & \alpha_1 & 0 & 0 & \cdot\\ \alpha_1 & \beta_1 & \alpha_2 & 0 & \cdot\\ 0 & \alpha_2 & \beta_2 & \alpha_3 & \cdot\\ \cdot & \cdot & \cdot & \cdot & \cdot\end{pmatrix} \qquad (\alpha_k>0). \tag{47}$$

The Jacobi matrix (47) defines a symmetrical operator $\dot{L}$ on the manifold of all the finite vectors of the space l_2 (that is, vectors with a finite number of non-vanishing coordinates); we denote by L the closure of $\dot{L}$. In contrast to the differential operator, the operator L is bounded, provided the coefficients of operation (46), i.e., the elements of the matrix (47), are bounded. We then have $\|L\|\leqslant \sup|\beta_k|+2\sup\alpha_k$ /6 (2)/.

A discrete analog of the one-dimensional differential operator of any order $2n$ is correspondingly the generalized Jacobi matrix, i.e., the infinite symmetric matrix (a_{jk}), where $a_{jk}=0$ for $|j-k|>n$.

Any self-adjoint extension of the operator L, defined by the matrix (47), is finite dimensional since $\mathrm{Def} L \leqslant 1$. The spectrum $C(\tilde{L})$, and the type of each point of $C(\tilde{L})$ do not depend therefore on the choice of the extension. Equally, the non-existence of an upper and lower bound to the spectrum does not depend on the extension.

Let L_0 be an operator defined by $L_0 y = Ly$ for all vectors $y \in \mathfrak{D}_L$ which satisfy, for a given natural number γ, the condition $y_\gamma = y_{\gamma+1} = 0$. We obtain a finite dimensional extension of the operator $\tilde{L}$ given by the formulas

$$\tilde{L} = \tilde{L}_0, \quad L_0 = L_1 \oplus L_\gamma,$$

where L_1 and L_γ are operators induced by the operator L_0 in the subspaces of the vectors $y = \{y_k\}_{k=0}^{\infty}$ of l_2. These operators satisfy the conditions $y_k = 0$ for $k \geqslant \gamma + 1$ and $y_k = 0$ for $k \leqslant \gamma$.

It follows from the decomposition principle that the spectrum $C(\tilde{L})$ does not depend on the values of any finite number of the elements of matrix (47). Furthermore, if l_1 and l_2 are two matrices of the form (47) such that the elements of the matrix $l_2 - l_1$ tend to zero, then $C(\tilde{L}_1) = C(\tilde{L}_2)$. In this case these propositions follow directly from Theorem 8 of H. Weyl, since under the given condition the operator $L_2 - L_1$ is completely continuous.

From the decomposition principle it also follows that the type of each point of the continuous part of the spectrum $C(\tilde{L})$, as well as the non-existence of a lower and upper bound to the spectrum $S(\tilde{L})$ do not depend on the values of a finite number of elements of matrix (47).

Introducing the integer-variable functions $N(m; \lambda_0, \delta)$, $N_1(m; \mu)$, and $N_2(m; \mu)$ for the number of eigenvalues of the square matrix of m-th order

$$\begin{pmatrix} \beta_0 & \alpha_1 & 0 & . & \\ \alpha_1 & \beta_1 & \alpha_2 & . & \\ & & & . & \\ & & & . & \\ 0 & 0 & 0 & . & \alpha_{m-1}\beta_{m-1} \end{pmatrix}, \tag{48}$$

lying respectively in the interval $(\lambda_0 - \delta, \lambda_0 + \delta)$, on the semi-axis $(-\infty, \mu)$, and on the semi-axis $(\mu, +\infty)$ as in No. 9, we obtain that these functions are monotonic increasing in the main and it is possible to prove theorems analogous to Theorems 26 and 27.

We continue our discussion of the oscillatory properties of the solution of the finite-difference second-order equation

$$-\alpha_{k+1} y_{k+1} - \alpha_k y_{k-1} + \beta_k y_k = \lambda y_k \qquad (y_{-1} = 0;\ k = 0, 1, 2, \ldots), \tag{49}$$

which is determined uniquely to within a constant factor.

We call the solution $y = \{y_k\}_{k=0}^{\infty}$ of equation (49) oscillatory and (49) itself an oscillatory equation, if the sequence of the coordinates y_k $(k = 0, 1, \ldots)$ contains an infinite number of alternations in sign. If this is not so, we call the solution non-oscillatory, and the equation a non-oscillatory equation.

Theorem 32. In order that equation (49) be oscillatory for $\lambda = \lambda_0$, it is necessary and sufficient that the part of the spectrum $S(\tilde{L})$ situated to the left of the point λ_0 be an infinite set.

Proof. We may assume without loss of generality that $\lambda_0 = 0$. If the set $(-\infty, 0) \cap S(\tilde{L})$ is infinite, then for an arbitrarily given N there exists a

principal minor of the form (48) situated in the left upper corner of the matrix (47) and having no less than N negative eigenvalues. From the Jacobi-Sylvester theorem, the number of negative eigenvalues of matrix (48) is equal to the number of alternations in sign in the sequence $(\zeta_k)_{k=0}^{m}$ ($\zeta_0 = 1$) of the principal minors of the matrix (48). Therefore the number of alternations in the sequence $\{\zeta_k\}_{k=0}^{\infty}$ of all the principal minors of the matrix (47) is infinite.

On the other hand (see for example /26/), expanding the determinant ζ_{k+1} with respect to the elements of the last line, we obtain the equality

$$\zeta_{k+1} = \beta_k \zeta_k - \alpha_k^2 \zeta_{k-1} \qquad (k = 2, 3, \ldots), \tag{50}$$

or putting

$$y_k = (\alpha_k \alpha_{k-1} \ldots \alpha_1)^{-1} \zeta_k \qquad (k = 1, 2, \ldots), \tag{51}$$

the equality

$$(-1)^{k+1} \alpha_{k+1} \alpha_k \ldots \alpha_1 y_{k+1} = \\ = (-1)^k \beta_k \alpha_k \alpha_{k-1} \ldots \alpha_1 y_k + (-1)^k \alpha_k^2 \alpha_{k-1} \ldots \alpha_1 y_{k-1},$$

or, finally

$$-\alpha_{k+1} y_{k+1} - \alpha_k y_{k-1} + \beta_k y_k = 0.$$

Setting

$$y_0 = 1, \tag{52}$$

we can directly verify the equalities

$$-\alpha_2 y_2 - \alpha_1 y_0 + \beta_1 y_1 = 0,$$
$$-\alpha_1 y_1 + \beta_0 y_0 = 0,$$

so that the sequence $\{y_k\}_{k=0}^{\infty}$, defined by equalities (51) and (52), is the solution of equation (49) for $\lambda = 0$.

Since $y_k \zeta_k \geqslant 0$, the solution $\{y_k\}_{k=0}^{\infty}$ has an infinite number of zeros, and the first part of the theorem is proved.

Conversely, if the solution $\{y_k\}_{k=0}^{\infty}$ of equation (49) for $\lambda = 0$ is oscillatory, then it follows from equality (51) that the sequence $\{\zeta_k\}_{k=1}^{\infty}$ of the principal minors contains an infinite number of alternations in sign. Using once more the Jacobi-Sylvester theorem, we obtain that when $m \to \infty$ then $N_1(m;\ 0) \to \infty$; hence it follows that the set of points of the negative part of the spectrum of the operator $\tilde{L}$ is infinite. Q. E. D.

If we discard in (49) the equation corresponding to the value $k = 0$, which plays the role of a boundary condition, then the resulting second-order finite-difference equation will have two linearly independent solutions. Since the fact that the set $(-\infty, 0) \cap S(\tilde{L})$ is infinite does not depend on the values of any finite number of elements of the matrix (47), then it follows from Theorem 32 that if one of the solutions of the second-order finite-difference equation (49) has an infinite number of alternations in sign, then any other solution of this equation possesses this property.

Some new results with regard to the spectrum of Jacobian matrices will be described further on, in Nos. 37 and 67.

§ 3. Multidimensional self-adjoint differential operators

15. Inclusion of multidimensional boundary-value problems in the general theory of symmetric operators. In the case of a multidimensional boundary-value problem, the simplest second-order differential operation* has the form

$$l[u] = -\Delta u + q(P)\,u, \tag{1}$$

where the function $q(P)$, called the potential, is assumed continuous in a bounded or unbounded domain* Ω of an m-dimensional Euclidian space $\mathfrak{E}_m$ (in particular it may be that $\Omega = \mathfrak{E}_m$). For $m = 3$, we write $\mathfrak{E}$ instead of $\mathfrak{E}_3$. The distance between two points P and Q of the space $\mathfrak{E}_m$ will be denoted by $|PQ|$ or $r(P, Q)$.

Confining ourselves in this and the following subsections to functions of three variables, we introduce the Hilbert space $\mathscr{L}_2(\Omega)$ of complex-valued point functions, which are measurable and whose modulus is square-integrable in the domain Ω, where the scalar product is defined by

$$(u, v) = \int_{\Omega} u(Q)\,\overline{v(Q)}\,d\omega. \tag{2}$$

Operation (1) will be called r e g u l a r, if the domain Ω is bounded and the function $q(P)$ is continuous up to the boundary of this domain. If at least one of these two conditions is not satisfied, operation (1) will be called s i n g u l a r.

We shall call the function $u(P)$ f i n i t e i n t h e d o m a i n Ω, if it identically vanishes outside some bounded domain lying wholly in Ω. The least closed domain outside which a given finite function identically vanishes will be called its s u p p o r t.

The inclusion of the multidimensional boundary-value problem in the general theory of symmetric operators is begun with the construction of the minimum operator L defined by operation (1), and the adjoint operator L^*.

The closure L of the symmetric operator L in $\mathscr{L}_2(\Omega)$, defined by operation (1) on the manifold of all twice continuously differentiable functions finite in Ω, will be called the m i n i m u m d i f f e r e n t i a l o p e r a t o r defined by operation (1) in the domain Ω.

To construct the operator L^* /31 (5)/ we first introduce the manifold of functions which are "potential" in Ω. This manifold is analogous to the manifold of functions of one variable having an absolutely continuous first derivative in the given interval.

We say that the function $u(P)$ is p o t e n t i a l in the domain Ω, if in each bounded domain $\omega \subset \Omega$ it may be written as

$$u(P) = -\frac{1}{4\pi}\int_{\omega} \frac{h(Q)}{r(P, Q)}\,d\omega_Q + \gamma(P) \qquad (P \in \omega), \tag{3}$$

where $h(Q) \in \mathscr{L}_2(\omega)$ and $\gamma(P)$ is a function harmonic in ω and belonging to $\mathscr{L}_2(\omega)$.

* Here and in what follows the restrictions imposed on the domains and their boundaries are not specified. We assume that these restrictions ensure the possibility of further constructions connected with the continuation of the functions beyond the boundary of their domain of definition. Apart from this these restrictions should be sufficient for the correctness of the classical boundary-value problems encountered in the book.

Here and throughout Nos. 15 — 20 we shall take the relation $\omega_1 \subset \omega_2$ between two domains ω_1 and ω_2 to imply that the domain ω_1 is bounded and lies wholly inside ω_2. Let the function $u(P)$ be twice continuously differentiable in the domain Ω. Choosing for any domain $\omega \subset \Omega$ a domain ω' which satisfies the relation $\omega \subset \omega' \subset \Omega$, we can represent the function $u(P)$ inside ω' as a sum of three terms

$$u(P) = -\frac{1}{4\pi}\int_{\omega}\frac{h(Q)}{r(P,Q)}\,d\omega_Q + \gamma_1(P) + \gamma_2(P)$$

with a volume mass density $h(Q)$ continuous up to the boundary of the domain ω'; hence, setting for $P \in \omega$

$$\gamma(P) = \gamma_1(P) + \gamma_2(P) - \frac{1}{4\pi}\int_{(\omega'-\omega)}\frac{h(Q)}{r(P,Q)}\,d\omega_Q,$$

we obtain representation (3).

Thus, any twice continuously differentiable function in the domain Ω is potential, but a potential function, being continuous by virtue of representation (3), is not in general twice continuously differentiable. It follows from representation (3) that in the case of a continuous distribution of masses $h(Q)$, a potential function will only have continuous first-order partial derivatives.

The set of all potential functions is obviously linear.

For any twice continuously differentiable function $u(P)$ representation (3) holds, in which

$$\Delta u(P) = h(P). \tag{4}$$

The following lemma together with equalities (3) and (4) allow us to extend the operation Δ to the manifold of all the potential functions in Ω.

Lemma 7. 1°. If the function $u(P)$ is potential in the domain Ω, then for a fixed domain $\omega \subset \Omega$, the components $h(P)$ and $\gamma(P)$ of the function $u(P)$ are uniquely determined by representation (3).

2°. A change of the domain ω does not change the values of the function $h(P)$ at points of the domain ω.

Proof. Assuming that some function has two representations of the form (3), we obtain

$$-\frac{1}{4\pi}\int_{\omega}\frac{g(Q)}{r(P,Q)}\,d\omega_Q + \vartheta(P) = 0.$$

To prove the first part of the lemma it is sufficient to show that $g(Q) = 0$.

Let us construct a sequence of functions $g_k(Q)$ which are continuously differentiable in ω and converge to $g(Q)$ in $\mathscr{L}_2(\omega)$. Let

$$u_k(P) = -\frac{1}{4\pi}\int_{\omega}\frac{g_k(Q)}{r(P,Q)}\,d\omega_Q + \vartheta(P). \tag{5}$$

Since the first term in (5) is the value of an operator bounded in $\mathscr{L}_2(\omega)$ on the function $g_k(Q)$, then $\lim_{k\to\infty} u_k = 0$ in the metric of $\mathscr{L}_2(\omega)$.

Further, for any function $v(P)$ twice continuously differentiable and finite in ω we have

$$(v, g) = \lim_{k\to\infty}(v, g_k) = \lim_{k\to\infty}(v, \Delta u_k) = \lim_{k\to\infty}(\Delta v, u_k) = 0. \tag{6}$$

Since the manifold of all the functions which are twice continuously differentiable and finite in ω is dense in $\mathscr{L}_2(\omega)$, it follows from (6) that $g=0$ in $\mathscr{L}_2(\omega)$, contrary to the assumption.

We now prove the second part of the lemma. We assume that representation (3) holds for $\omega=\omega_j$, $h=h_j$, $\gamma=\gamma_j$ $(j=1, 2)$. Associating in these two representations the integrals over $\omega_1-(\omega_1\cap\omega_2)$ and $\omega_2-(\omega_2\cap\omega_1)$ with γ_1 and γ_2, respectively, we obtain from the first part of the lemma that $g_1(P)=g_2(P)$ for $P\in\omega_1\cap\omega_2$. Q. E. D.

According to Lemma 1, to any function $u(P)$ which is potential in Ω there corresponds in representation (3) a single function $h(P)$, defined in Ω and belonging to $\mathscr{L}_2(\omega)$ for any $\omega\subset\Omega$. Expanding the operation Δ, we assume that for any function which is potential in Ω

$$\Delta u(P)=h(P).$$

We now consider the characterization of the operator L^*.

Theorem 33. The operator L^* is determined by operation (1) on the manifold of all the functions $u(P)$ of $\mathscr{L}_2(\Omega)$ which are potential in Ω, and for which $l[u]\in\mathscr{L}_2(\Omega)$.

Proof. (see /31 (5)/). Let M deonote the operator determined by operation (1) on the manifold of all the functions v of $\mathscr{L}_2(\Omega)$ which are potential in Ω, and for which $l[v]\in\mathscr{L}_2(\Omega)$. We shall show that $L^*=M$.

Let $v\in\mathfrak{D}_M$, and $u(P)$ a function which is finite and twice continuously differentiable in Ω. We have

$$(l[u], v)=-(\Delta u, v)+(qu, v)=-\lim_{k\to\infty}(\Delta u, v_k)+(u, qv)=$$
$$=-\lim_{k\to\infty}(u, \Delta v_k)+(u, qv)=-(u, \Delta v)+(u, qv)=(u, l[v]),$$

where the sequence v_k is constructed similarly to the sequence u_k in (5).

Thus, for $v\in\mathfrak{D}_M$ we have for any function u twice continuously differentiable and finite in Ω that

$$(l[u], v)=(u, l[v]),$$

hence it follows that $M\subset L^*$.

Conversely, let $v\in\mathfrak{D}_{L^*}$, and let $u(P)$ be a function twice continuously differentiable and finite in some domain $\omega\subset\Omega$. We have

$$(l[u], v)=(u, v^*)$$

or

$$(\Delta u, v)=(u, qv-v^*).$$

Therefore, if we set for $P\in\omega$ and $h=qv-v^*$

$$v_0(P)=-\frac{1}{4\pi}\int_\omega\frac{h(Q)}{r(P, Q)}d\omega_Q,$$

we obtain

$$(\Delta u, v)=(u, \Delta v_0)=\lim_{k\to\infty}(u, \Delta v_k)=\lim_{k\to\infty}(\Delta u, v_k)=(\Delta u, v_0), \qquad (7)$$

where the sequence v_k is constructed similarly to the sequence u_k in (5).

Thus from (7) we have

$$(\Delta u, v-v_0)=0,$$

That is, for a given function $\eta(P)=v(P)-v_0(P)$ which is measurable in ω we have

$$\int_{\omega}\eta(Q)\,\Delta u\,(Q)\,d\omega=0. \tag{8}$$

for any $u(P)$ twice continuously differentiable and finite in ω.

From equality (8) together with the well-known lemma of S. L. Sobolev* /90 (1)/ it follows that the function $\eta(P)$ coincides almost everywhere in ω with some harmonic function $\gamma(P)$. Thus the function v of $\mathfrak{D}_{L^*}$ may be written in the form

$$v(P)=v_0(P)+\gamma(P) \qquad (P\in\omega),$$

hence the function $v(P)$ is potential in the domain ω. Since ω is an arbitrary bounded domain contained wholly inside Ω, the function $v(P)$ is potential in Ω.

Furthermore, $v\in\mathscr{L}_2(\Omega)$ and

$$l[v]=l[v_0]+l[\gamma]=-\Delta v_0+q(v_0+\gamma)=v^*\in\mathscr{L}_2(\Omega),$$

so that

$$L^*\subset M$$

which completes the proof of the theorem.

According to the general theory of symmetric operators, the domain of definition of the operator L^* is given by the formula

$$\mathfrak{D}_{L^*}=\mathfrak{D}_L\dotplus\mathfrak{N}_\lambda\dotplus\mathfrak{N}_{\bar\lambda} \qquad (\operatorname{Im}\lambda\neq 0),$$

and the domain of definition of any self-adjoint extension $\tilde L$ of the operator L is given by the formula

$$\mathfrak{D}_{\tilde L}=\mathfrak{D}_L\dotplus\{\mathfrak{N}_\lambda+U\mathfrak{N}_\lambda\} \qquad (\operatorname{Im}\lambda\neq 0),$$

where $\mathfrak{N}_\lambda$ is, for any λ, a subspace of the solutions of the equation

$$L^*g-\lambda g=0, \tag{9}$$

and U is a unitary operator mapping $\mathfrak{N}_\lambda$ onto $\mathfrak{N}_{\bar\lambda}$.

In order to avoid the need of introducing generalized solutions of equation (9), we shall in the following consider the function $q(P)$ continuously differentiable in the domain Ω.

With this assumption, by virtue of some well-known theorems of potential theory (see, for example, /91/), the subspaces $\mathfrak{N}_\lambda$ and $\mathfrak{N}_{\bar\lambda}$ consist only of twice continuously differentiable functions. We may therefore introduce the unclosed operators $\tilde L'(\tilde L'\subset\tilde L)$ and $M'(M'\subset L^*)$ with the domains of definition

$$\mathfrak{D}_{M'}=\mathfrak{D}_L\dotplus\mathfrak{N}_\lambda\dotplus\mathfrak{N}_{\bar\lambda}$$

and

$$\mathfrak{D}_{\tilde L'}=\mathfrak{D}_L\dotplus\{\mathfrak{N}_\lambda+U\mathfrak{N}_\lambda\}. \tag{10}$$

These domains of definition consist of twice continuously differentiable functions only. The operators L^* and $\tilde L$, being obviously the closures ot the operators M' and $\tilde L'$ respectively, are uniquely determined by these operators.

Formula (10), which determines the choice of the manifold $\mathfrak{D}_{\tilde L'}$ as part of the manifold $\mathfrak{D}_{M'}$, does not impose any restrictions on the behavior of the

* This is a particular case of the lemma which was considered by H. Weyl /20 (4)/. H. Weyl's proof is reproduced at the end of this subsection.

functions of $\mathfrak{D}_{M'}$ at internal points of the domain Ω. In fact, if $u(P) \in \mathfrak{D}_{\tilde{L}'}$, and $v(P)$ is any function of $\mathfrak{D}_{M'}$ coinciding with $u(P)$ outside some domain $\omega \subset \Omega$, then $v(P)$ also belongs to $\mathfrak{D}_{\tilde{L}'}$, because it is the sum of the two functions $u(P)$ of $\mathfrak{D}_{\tilde{L}'}$, and $v(P) - u(P)$ of $\mathfrak{D}_{L} \subset \mathfrak{D}_{\tilde{L}'}$ which is finite in Ω.

Thus the conditions which under a definite choice of the operator U isolate the class of functions $\mathfrak{D}_{\tilde{L}'}$ of $\mathfrak{D}_{M'}$ do not restrict in the above sense the behavior of the functions of $\mathfrak{D}_{M'}$ in any domain $\omega \subset \Omega$. Therefore these conditions implicitly specified by the operator U in formula (10) are called boundary conditions. The classical boundary conditions of the form $\left[\frac{\partial u}{\partial n} + \rho u\right]_{p \in S} = 0$ are a particular case of the boundary conditions we consider here.

The problem of the explicit form of the boundary conditions, and of the solution of the corresponding boundary-value problems for regular differential operations has been considered in a number of papers by M.I. Vishik (see /24/).

If operation (1) is regular, and the boundary conditions determined by the operator U are such that the spectrum $S(\tilde{L})$ is semi-bounded from below and contains only eigenvalues of finite multiplicity with a single limit point $\lambda = +\infty$, then we say that these boundary conditions are r e g u l a r. If this is not so, the boundary conditions for a regular operation will be called singular.

The operator $\tilde{L}$ will be called r e g u l a r, if operation (1) and the corresponding boundary conditions are regular. If even one of these two conditions is not fulfilled the operator $\tilde{L}$ is said to be s i n g u l a r.

Let $\Omega[\sigma, \zeta]$ denote that part of the space $\mathfrak{S}$ which is situated between the closed surfaces σ and ζ. ζ may be at infinity, in which case $\Omega[\sigma, \zeta]$ is that part of the space $\mathfrak{S}$ situated outside the surface σ.

With $\Omega = \Omega[\sigma, \zeta]$, the boundary conditions determined by the operator U are said to be s e p a r a t e d if from $u(P) \in \mathfrak{D}_{\tilde{L}'}$ it follows that $v(P) \in \mathfrak{D}_{\tilde{L}'}$, where $v(P)$ is any function of $\mathfrak{D}_{M'}$ which coincides with $u(P)$ near the boundary of ζ and is zero near the boundary of σ, or equivalently, coincides with $u(P)$ near σ and is zero near ζ. The separated boundary conditions are analogous to the Sturm conditions of the one-dimensional case.

For any two functions $u, v \in \mathfrak{D}_{\tilde{L}'}$ we have the equality

$$(\Delta u, v) = (u, \Delta v),$$

which shows that the limit

$$\lim_{\sigma' + \zeta'} \int \left\{ u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right\} ds$$

exists and is equal to zero no matter how σ' approaches σ, and ζ' approaches ζ.

A necessary and sufficient condition for the separability of the boundary conditions determining the operator $\tilde{L}$ is that for any two functions $u, v \in \mathfrak{D}_{\tilde{L}'}$ the following limit relations are satisfied

$$\lim_{\sigma' \to \sigma} \int_{\sigma'} \left\{ u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right\} ds = 0,$$

$$\lim_{\zeta' \to \zeta} \int_{\zeta'} \left\{ u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right\} ds = 0. \qquad (11)$$

The above condition is obviously necessary. To show that it is sufficient we assume that the function $w(P) \in \mathfrak{D}_{M'}$ vanishes near σ and coincides with

some function $v(P) \in \mathfrak{D}_{\tilde{L}'}$ near ζ. We have to show that $w \in \mathfrak{D}_{\tilde{L}'}$. Since the operator $\tilde{L}$ is self-adjoint, it is sufficient to show that the following relation holds for every function $u \in \mathfrak{D}_{\tilde{L}'}$

$$(l[u], w) = (u, l[w]).$$

The above equality is equivalent to

$$\lim_{\substack{\sigma' \to \sigma \\ \zeta' \to \zeta}} \int_{\sigma' + \zeta'} \left\{ u \frac{\partial w}{\partial n} - w \frac{\partial u}{\partial n} \right\} ds = 0,$$

which is satisfied by virtue of (11), since near σ we have $w = \frac{\partial w}{\partial n} = 0$, and near ζ we have $w = v$, $\frac{\partial w}{\partial n} = \frac{\partial v}{\partial n}$.

We now introduce the concept of equivalence of boundary conditions.

Let $\tilde{L}_1$ and $\tilde{L}_2$ be self-adjoint operators generated by the same operation (1) in the domains $\Omega[\sigma_1, \zeta]$ and $\Omega[\sigma_2, \zeta]$, respectively. The boundary conditions for the operators $\tilde{L}_1$ and $\tilde{L}_2$ on the surface ζ (or at infinity, if ζ is not stipulated) are said to be equivalent (identical) if the fact that a function of $\mathfrak{D}_{M'}$ which is zero inside a closed surface σ (including both σ_1 and σ_2) belongs to $\mathfrak{D}_{\tilde{L}_1'}$ implies that this function also belongs to $\mathfrak{D}_{\tilde{L}_2'}$. Note that when this requirement is satisfied, then if a function of $\mathfrak{D}_{M'}$ vanishing inside a closed surface σ belongs to $\mathfrak{D}_{\tilde{L}_2'}$ it also belongs to $\mathfrak{D}_{\tilde{L}_1'}$.

A necessary and sufficient condition for the equivalence of the boundary conditions on ζ (or at infinity) is that for any pair of functions $u \in \mathfrak{D}_{\tilde{L}_1'}$ and $v \in \mathfrak{D}_{\tilde{L}_2'}$ the limit relation (11) is satisfied for $\zeta' \to \zeta$ (or for ζ' approaching infinity, if ζ is not stipulated).

To conclude this subsection we give H. Weyl's proof of the lemma which was used in the proof of Theorem 33.

Lemma 8 /20 (4)/. If for a given function $\eta(P)$ measurable in the domain ω equality (8) holds for any function $u(P)$ twice continuously differentiable and finite in the domain ω, then the function $\eta(P)$ coincides almost everywhere in ω with some harmonic function.

Proof. Let P_0 be an arbitrary point of ω, and d its distance from the boundary of the domain ω. Clearly it is sufficient to show that almost everywhere in a spherical neighborhood of radius ρ of the point P_0, the function $\eta(P)$ coincides with some harmonic function $\gamma(P)$. The proof of this statement is based on the use of equality (8) for the special system of functions

$$u(Q) = u_1(Q) - u_2(Q). \tag{12}$$

To construct the function $u_1(Q)$ we start from the potential $u_1^0(Q, P; \rho)$ of the sphere $r(PM) = \rho$ centered at the point P and charged with a constant surface density $\sigma_1(M) = \frac{1}{4\pi\rho^2}$. Obviously,

$$u_1^0(Q, P; \rho) = \begin{cases} \frac{1}{\rho} & \text{for } r(P, Q) < \rho, \\ \frac{1}{r(P, Q)} & \text{for } r(P, Q) \geqslant \rho. \end{cases}$$

Replace the sphere $r(P, M) = \rho$ by a sphere $r(P, M) < c$ with a volume charge density at the point M equal to

$$\sigma_1(M) \rho^2 \mu_1(\rho),$$

where $\rho = r(P, M)$. The function $\mu_1(\rho)$ will be defined later.

The potential of this sphere is given by

$$u_1(Q) = u_1(Q, P; c) = \int_0^c u_1^0(Q, P; \rho) \mu_1(\rho) \rho^2 d\rho. \tag{13}$$

To construct the function $u_2(Q)$ we start from the function

$$u_2^0(Q, P; R) = \begin{cases} \dfrac{1}{r(P_0, P) \cdot r(P^*, Q)} & \text{for } r(P_0, Q) < R \\ & (r(P_0, P)\, r(P_0, P^*) = R^2), \\ \dfrac{1}{r(P, Q)} & \text{for } r(P_0, Q) \geqslant R, \end{cases}$$

which may be considered as the potential of a simple spherical layer $r(P_0, M) = R$ with a surface charge density at the point M equal to

$$\sigma_2(M) = \frac{1}{R} \frac{R^2 - r^2(P_0, P)}{r^2(P, M)}.$$

Let us replace the spherical layer by a spherical annulus $a < r(P_0, M) < b$, which is charged with a volume density

$$\sigma_2(M)\, \mu_2(R)\, R,$$

where $R = r(P_0, M)$, and the function $\mu_2(R)$ will be defined below.

We assume in this connection that the spherical shell $a \leqslant r(P_0, M) < b$ lies entirely in the domain ω, and the sphere $r(P_0, M) = a$ contains the sphere $r(P, M) = c$. The potential of the spherical annulus is given by

$$u_2(Q) = u_2(Q, P; a, b) = \int_a^b u_2^0(Q, P; R)\, \mu_2(R)\, R\, dR. \tag{14}$$

We now assume that the functions $\mu_1(\rho)$ and $\mu_2(\rho)$ fulfil the conditions

$$\int_0^c \mu_1(\rho)\, \rho^2\, d\rho = 1, \quad \int_a^b \mu_2(R)\, R\, dR = 1,$$

so that the functions $u_1(Q)$ and $u_2(Q)$ are each equal to $\dfrac{1}{r(P, Q)}$ outside the sphere $r(P_0, M) = b$. Furthermore, we shall consider the functions $\mu_1(\rho)$ and $\mu_2(\rho)$ as continuously differentiable and satisfying the conditions

$$\mu_1(c) = \mu_1'(c) = \mu_2(a) = \mu_2'(a) = \mu_2(b) = \mu_2'(b) = 0.$$

With this choice of the functions μ_1 and μ_2, the charge densities of the sphere $r(P, M) \leqslant c$ and of the spherical annulus $a \leqslant r(P_0, M) \leqslant b$, which set up the potentials (13) and (14), are continuous together with their partial derivatives in the domain ω. The functions $u_1(Q)$ and $u_2(Q)$ will therefore be twice continuously differentiable. They then satisfy the Poisson equation. Thus, the function $u(Q)$ constructed according to formula (12) is twice continuously differentiable and finite in ω. Substituting the function $u(Q)$ in equality (8) and using the Poisson equation, we obtain

$$\frac{\iiint\limits_{r(P, Q) < c} \mu_1(Q)\, \eta(Q)\, d\omega_Q}{\int_0^c \mu(\rho)\, \rho^2\, d\rho} = \frac{\iiint\limits_{a < r(P_0, Q) < b} \dfrac{r^2(P_0, Q) - r^2(P_0, P)}{r^3(Q, P)}\, \mu_2(Q)\, \eta(Q)\, d\omega_Q}{\int_a^b \mu_2(R)\, R\, dR}. \tag{15}$$

The right-hand side of equality (15) obviously does not depend on c. Neither does it depend on a and b, as the left-hand side of (15) does not contain a and b. Since the expression

$$\frac{r^2(P_0, Q) - r^2(P_0, P)}{r^3(Q, P)}$$

is a harmonic function of the point P, then by virtue of the superposition principle the right-hand side of equality (15) is some harmonic function $\gamma(P)$ in the domain ω.

Taking in (15) the limit for $c \to 0$, we obtain for almost all $P \in \omega$ the equality

$$4\pi\eta(P) = \gamma(P),$$

Q. E. D.

We note that the proof of Lemma 8 would be appreciably simplified, if the function $u(P)$ could a priori be considered continuous /89/. In connection with Lemma 8, see also /29/.

16. On the defect index of partial differential operators. In the one-dimensional case this problem was not given special consideration, since there the defect numbers could not be infinite. Since the investigated properties of the spectrum are the same for all the finite dimensional self-adjoint extensions of a given operator, it was unnecessary to consider the theorems which enable us to estimate the value of the defect numbers from the behavior of the coefficients of the differential operation.

In the multi-dimensional case the situation is different. Here the defect number of the minimal operator L is equal to the dimension of the linear manifold of all the solutions of the equation

$$-\Delta u + qu - \lambda u = 0 \qquad (\operatorname{Im}\lambda \neq 0),$$

belonging to $\mathscr{L}_2(\Omega)$, and this manifold may be infinite dimensional.

In particular, for the case of the regular operation (1), we always have $\operatorname{Def} L = \infty$, and in the singular case we have

$$0 \leqslant \operatorname{Def} L \leqslant \infty. \tag{16}$$

For $\Omega = \mathfrak{E}$ it is easy to find examples where the defect number takes on the extreme values indicated by inequality (16), but examples where $0 < \operatorname{Def} L < \infty$ are not known.

We now give two criteria for the vanishing of the defect number of the operator L. The first criterion (Theorem 34) is due to T. Carleman and K. Friedrichs, and the second (Theorem 35) was first established by A.Ya. Povzner.

The following lemma was established in /23 (1)/. The proof of this lemma given below is due mainly to F.S. Rofe-Beketov.

Lemma 9 /23 (1)/. If a potential is bounded from below, then $u \in \mathscr{L}_2(\mathfrak{E})$ *and* $l[u] \in \mathscr{L}_2(\mathfrak{E})$ *imply that* $|\nabla u| \in \mathscr{L}_2(\mathfrak{E})$.

Proof. From Green's first formula we have

$$\int\limits_{|OP|=R} u \frac{\partial u}{\partial R}\, ds = \int\limits_{|OP|<R} |\nabla u|^2\, d\omega + \int\limits_{|OP|<R} u\, \Delta u\, d\omega. \tag{17}$$

On the other hand, from $u \in \mathscr{L}_2(\mathfrak{E})$ and $l[u] \in \mathscr{L}_2(\mathfrak{E})$ it follows that as $R \to \infty$ there exists a finite limit to the sum

$$-\int\limits_{|OP|<R} u\, \Delta u\, d\omega + \int\limits_{|OP|<R} qu^2\, d\omega, \tag{18}$$

hence, since the potential is bounded from below, it follows that for $R \to \infty$ the first term in (18) tends either to a finite limit or to minus infinity. But then the right-hand side of equality (17), and consequently the function

$$F(R) = \int\limits_{|OP|=R} u \frac{\partial u}{\partial R}\, ds$$

have either a finite limit or tends to $+\infty$ as $R \to \infty$. Therefore, to prove the lemma it is sufficient to show that

$$\lim_{R\to\infty} F(R) = +\infty, \tag{19}$$

is impossible.

Introducing the function

$$J(R) = \int\limits_{|OP|=R} |u|^2 \, ds$$

and setting

$$d\sigma = \sin\theta \, d\varphi \, d\theta,$$

we obtain

$$\frac{dI}{dR} = \frac{d}{dR}\left\{ R^2 \int\limits_{|OP|=R} u^2 \, d\sigma \right\} =$$

$$= 2R \int\limits_{|OP|=R} u^2 \, d\sigma + 2R^2 \int\limits_{|OP|=R} u \frac{\partial u}{\partial R} \, d\sigma =$$

$$= \frac{2}{R} \int\limits_{|OP|=R} u^2 \, ds + 2F(R). \tag{20}$$

It can be seen from relation (20) that, in conjunction with (19), we also have

$$\lim_{R\to\infty} J'(R) = +\infty,$$

and a fortiori

$$\lim_{R\to\infty} J(R) = +\infty,$$

contradicting $u(P) \in \mathscr{L}_2(\mathfrak{E})$, since clearly

$$\int\limits_{\mathfrak{E}} |u|^2 \, d\omega = \int\limits_0^\infty J(R) \, dR.$$

The lemma is therefore proved.

Theorem 34 /47/. If the potential $q(P)$ is bounded from below, the defect numbers of the corresponding operator L are equal to zero, that is $L = L^*$.

Proof. It is sufficient to show that the operator L^* is symmetric, i.e.,

$$\int\limits_{\mathfrak{E}} \{u\Delta v - v\Delta u\} \, d\omega = 0 \tag{21}$$

for any pair of functions $u, v \in \mathfrak{D}_{L^*}$. This equality is obviously equivalent to the limit relation

$$\lim_{R_n\to\infty} \int\limits_{|OP|=R_n} \left\{ u \frac{\partial v}{\partial R} - v \frac{\partial u}{\partial R} \right\} ds = 0. \tag{22}$$

If the functions u and v belong to $\mathfrak{D}_{L^*}$, then from Lemma 9, we have

$$|\nabla u| \in \mathscr{L}_2(\mathfrak{E}), \quad |\nabla v| \in \mathscr{L}_2(\mathfrak{E})$$

and a fortiori

$$\frac{\partial u}{\partial R} \in \mathscr{L}_2(\mathfrak{E}), \quad \frac{\partial v}{\partial R} \in \mathscr{L}_2(\mathfrak{E}).$$

Thus,

$$\int\limits_{\mathfrak{E}} |u| \left| \frac{\partial v}{\partial R} \right| d\omega < \infty,$$

that is,

$$u \frac{\partial v}{\partial R} \in \mathscr{L}_1(\mathfrak{E}) \tag{23}$$

and, since

$$v\frac{\partial u}{\partial R}\in\mathscr{L}_1(\mathfrak{E}),$$

then

$$u\frac{\partial v}{\partial R}-v\frac{\partial u}{\partial R}\in\mathscr{L}_1(\mathfrak{E}),$$

that is

$$\int_{\mathfrak{E}}\left|u\frac{\partial v}{\partial R}-v\frac{\partial u}{\partial R}\right|d\omega<\infty$$

or

$$\int_0^\infty dR\int_{|OP|=R}\int\left|u\frac{\partial v}{\partial R}-v\frac{\partial u}{\partial R}\right|ds<\infty. \tag{24}$$

From inequality (24) if follows that there exists a sequence $R_n\to\infty$, for which the limit relation (22) holds. Q. E. D.

T. Carleman's proof of Theorem 34 was based on his Lemma on the existence, for any twice continuously differentiable function $u(P)\in\mathscr{L}_2(\mathfrak{E})$, of a sequence of spheres $|OP|=R_n$, on which the limit relation (26) is satisfied (see below).

Obviously R_n may be replaced by R in (22). This follows from the existence of the integral in the left-hand side of (21) for any u and v of $\mathfrak{D}_{L^*}$.

Note that for $u\in\mathfrak{D}_{L^*}$ it follows from relation (23) alone that

$$\int_0^\infty dR\int_{|OP|=R}\int\left|u\frac{\partial v}{\partial R}\right|ds<\infty$$

and also, for some sequence $R_n\to\infty$,

$$\lim_{R_n\to\infty}\int_{|OP|=R_n}u\frac{\partial v}{\partial R}\,ds=0. \tag{25}$$

In particular, for $v=u\in\mathfrak{D}_L$, relation (25) implies that the function

$$S(R)=\int_{|OP|=R}u\frac{\partial u}{\partial R}\,ds$$

tends to zero as $R_n\to\infty$, and since the existence of the limit of $S(R)$ for $R\to\infty$ follows from (17), then

$$\lim_{R\to\infty}\int_{|OP|=R}u\frac{\partial u}{\partial R}\,ds=0. \tag{26}$$

Now, according to Lemma 9 we obtain that the value of the quadratic functional $(Lu,\ u)$ for any function $u\in\mathfrak{D}_L$ is given by

$$(Lu,\ u)=\int_{\mathfrak{E}}|\nabla u|^2\,d\omega+\int_{\mathfrak{E}}q\,|u|^2\,d\omega, \tag{27}$$

which in the case of a potential unbounded from below is true, in general, only for finite functions of $\mathfrak{D}_L$.

It was noted by F.S. Rofe-Beketov that under certain restrictions as to the form of the boundary and of the boundary conditions Lemma 9, and consequently also Theorem 34, go over to the case $\Omega\neq\mathfrak{E}$.

To show this, we denote by S an unclosed surface dividing the space $\mathfrak{E}$ into two parts Ω and $\mathfrak{E}-\Omega$. The part Ω is called "internal". Let σ_R be that part of the sphere $|OP|=R$ lying in Ω, S_R that part of the surface S enclosing together with σ_R a finite part Ω_R of the domain Ω. Let n_P denote the external normal to S at the point P. Furthermore let the operator L be the closure in $\mathscr{D}_2(\Omega)$ of the operator defined by operation (1) on the twice continuously differentiable functions $u(P)$ which vanish identically for $|OP|>R_u$ and satisfy on S one of the three boundary conditions

$$u(P)|_{P\in S}=0, \tag{28}$$

$$\frac{\partial u}{\partial n}\Big|_{P\in S}=0, \tag{29}$$

or

$$\frac{\partial u}{\partial n}+pu\Big|_{P\in S}=0, \tag{30}$$

where

$$p=p(P)\geqslant 0.$$

It can be shown that the operator L^* will then be defined by operation (1) on all the potential functions $u(P)$ belonging, together with $l[u]$, to the space $\mathscr{D}_2(\Omega)$ and satisfying respectively one of the conditions (28), (29), or (30).

Now instead of (17) we have

$$\int_{\sigma_R} u\frac{\partial u}{\partial R}\,ds=\int_{\Omega_R}|\nabla u|^2\,d\omega+\int_{\Omega_R}u\Delta u\,d\omega-\int_{S_R}u\frac{\partial u}{\partial n}\,ds \tag{31}$$

and, setting

$$F(R)=\int_{\sigma_R}u\frac{\partial u}{\partial R}\,ds,\quad J(R)=\int_{\sigma_R}u^2\,ds,$$

we obtain instead of (20) the equality

$$\begin{aligned}\frac{dJ}{dR}&=2R\int_{\sigma_0(R)}u^2\,d\sigma+2R^2\int_{\sigma_0(R)}u\frac{\partial u}{\partial R}\,d\sigma-R\int_{C_0(R)}\operatorname{ctg}(n,R)\,u^2\,dl_0=\\&=\frac{2}{R}\int_{\sigma_R}u^2\,d\sigma-\int_{C_R}\operatorname{ctg}(n,R)\,u^2\,dl+2F(R),\end{aligned} \tag{32}$$

where $\sigma_0(R)$ is the central projection, from the origin of the coordinates, of the surface σ_R on the surface of the unit sphere $|OP|=1$, C_R is the intersection line of σ_R with S_R, $C_0(R)$ is its central projection on the sphere $|OP|=1$, and dl and dl_0 are the corresponding arc differentials.

In the case of boundary condition (28), relations (31) and (32) coincide with relations (17) and (20). In this case the proofs of Lemma 9 and Theorem 34 do not differ from those given above. In the case of boundary conditions (29) or (30) the proofs are similar, provided the surface is subjected to the requirement that

$$\operatorname{ctg}(n,R)\leqslant 0. \tag{33}$$

The above inequality means that when R increases the central projection of σ_R on the unit sphere $|OP|=1$ is not reduced. It is sufficient for our purpose to consider assumption (33) as satisfied for all the points $P\in S$ situated outside the sphere $|OP|=R_0$.

The value of the quadratic functional will be given by (27) when boundary conditions (28) and (29) hold. For boundary condition (30) we have, for any function $u \in \mathfrak{D}_L$,

$$(Lu,\ u) = \int_{\Omega} |\nabla u|^2 d\omega + \int_{\Omega} q|u|^2 d\omega + \int_{S} \rho |u|^2 ds.$$

The following proposition tentatively introduced by the author and established for the first time by A. Ya. Povzner (see /81/), shows that Theorem 34 remains valid when the requirement of the boundedness from below of the potential is replaced by a less restrictive assumption on the boundedness from below of the operator L. The following proof of this proposition is due to E. Wienholtz /22/.

Theorem 35 /81/. If the operator L defined by operation (1) in $\mathscr{L}_2(\mathfrak{G})$, is bounded from below, then it is self-adjoint.

Proof. We may assume, without loss of generality, that for any $u \in \mathfrak{D}_L$ we have

$$\int_{\mathfrak{G}} (-\Delta u + qu)\bar{u}\, d\omega > \int_{\mathfrak{G}} |u|^2 d\omega. \tag{34}$$

It is sufficient to show that under condition (34) the equation

$$-\Delta v + qv = 0 \tag{35}$$

cannot have non-trivial solutions in $\mathscr{L}_2(\mathfrak{G})$.

Assuming that v is the solution of equation (35), we set

$$u_R(P) = \theta_R(P)\, v(P), \tag{36}$$

where

$$\theta_R(P) = \theta_R(|OP|) = \theta\left(\frac{|OP|}{R}\right),$$

and $\theta(r)$ is a sufficiently smooth finite function of an argument $r \geqslant 0$, satisfying the conditions

$$\theta(r) = \begin{cases} 1 & \left(0 \leqslant r \leqslant \frac{1}{2}\right), \\ 0 & (r > 1). \end{cases}$$

It is obvious that the finite function $u_R(P)$ defined by equality (36) belongs to $\mathfrak{D}_L$. It therefore satisfies relation (34), that is

$$\int_{\mathfrak{G}} \theta_R^2 v^2\, d\omega < \int_{\mathfrak{G}} (-\Delta u_R + q u_R)\, u_R\, d\omega,$$

but

$$-\Delta u_R + q u_R = (-\Delta v + qv)\,\theta_R - 2\nabla v \cdot \nabla \theta_R - v\Delta\theta_R = \\ = -2\nabla v \cdot \nabla\theta_R - v\Delta\theta_R,$$

so that

$$\int_{\mathfrak{G}} \theta_R^2 v^2\, d\omega < -\int_{\mathfrak{G}} \{2\nabla v \cdot \nabla\theta_R + v\Delta\theta_R\}\, v\theta_R\, d\omega. \tag{37}$$

On the other hand,

$$2\int_{\mathfrak{G}} \nabla v \cdot \nabla\theta_R v\theta_R\, d\omega = \frac{1}{2}\int_{\mathfrak{G}} \nabla v^2 \cdot \nabla\theta_R^2\, d\omega = -\frac{1}{2}\int_{\mathfrak{G}} v^2 \Delta\theta_R^2\, d\omega = \\ = -\int_{\mathfrak{G}} v^2(\theta_R\Delta\theta_R + |\nabla\theta_R|^2)\, d\omega,$$

so that inequality (37) takes the form

$$\int_{\mathfrak{G}} \theta_R^2 v^2 \, d\omega < \int_{\mathfrak{G}} v^2 |\nabla \theta_R|^2 \, d\omega,$$

that is

$$\int_{|OP| < \frac{R}{2}} v^2 \, d\omega > \frac{C}{R^2} \int_{|OP| < R} v^2 \, d\omega. \tag{38}$$

If we now assume that the solution $v(P)$ belongs to $\mathscr{L}_2(\mathfrak{G})$, then from inequality (38) it follows that for $R \to \infty$, $v(P) \equiv 0$. Q. E. D.

Theorem (34) has also been generalized by D. Sears /87/ for the case of potentials $q(P)$ bounded from below by a function $\mu(r)$ which tends not too rapidly to minus infinity for $r \to \infty$.

17. Decomposition in multidimensional domains. Let, as before, $\Omega = \Omega[\sigma, \zeta]$ be that part of the space $\mathfrak{G}$ which is situated between the closed surfaces σ and ζ. In particular, when the surface σ is not stipulated, Ω is that part of the space $\mathfrak{G}$ which lies inside ζ. In the absence of the surface ζ, the domain Ω is that part of the space $\mathfrak{G}$ which lies outside σ. In the absence of both boundaries $\Omega = \mathfrak{G}$.

Let Γ denote an arbitrary closed surface which divides the domain Ω into two parts $\omega = \Omega[\sigma, \Gamma]$ and $\Omega_\Gamma = \Omega[\Gamma, \zeta]$. Furthermore, slightly modifying the similar construction of No. 7, we denote by L_0 the closure of the symmetric operator $\dot{L}_0$ defined by the equality

$$\dot{L}_0 u = \tilde{L} u$$

for all functions $u \in \mathfrak{D}_L$ which vanish identically near the decomposition surface Γ.

The symmetric operator L_0 is the orthogonal sum of the minimal differential operator L_ω and of the symmetric operator L_Γ, induced by the operator $\tilde{L}$ in $\mathscr{L}_2(\omega)$ and $\mathscr{L}_2(\Omega_\Gamma)$ respectively. Thus the relations

$$\tilde{L} = L_0, \quad L_0 = L_\omega \oplus L_\Gamma \tag{39}$$

define the decomposition of the operator $\tilde{L}$, but in contrast to the one-dimensional case, this decomposition is infinite dimensional. We cannot, therefore, justify the relation

$$C(\tilde{L}) = C(L_0) \tag{40}$$

by using the general theorems of No. 2.

Thus, the decomposition principle, consisting of the replacement of the operator $\tilde{L}$ by the "decomposed" operator L_0 and the use of relation (40), requires new substantiation. This is the purpose of the present subsection.

The proof of relation (40) for multidimensional differential operators is based on the following lemma.

Lemma 10 /31 (5)/. If $\lambda \in C(\tilde{L})$ and $\omega \subset \Omega$, there exists a characteristic sequence $\{v_n\}$ for λ which satisfies the condition

$$v_n(P) \equiv 0 \qquad (P \in \omega; \quad n = 1, 2, \ldots).$$

Proof. For $\lambda \in C(\tilde{L})$ there exists in $\mathfrak{D}_L$ an orthonormalized characteristic sequence $\{f_n\}$. On the other hand, since the operator $\tilde{L}$ is the closure of the

operator $\tilde{L}'$, there exists in $\mathfrak{D}_{\tilde{L}'}$ a sequence $\{\tilde{\varphi}_n\}$, such that

$$\lim_{n\to\infty}(f_n-\varphi_n)=0$$

and

$$\lim_{n\to\infty}(\tilde{L}f_n-\tilde{L}\varphi_n)=0.$$

From these limit relations it follows that the sequence $\{\varphi_n\}$ is also characteristic for the given λ and $\lim_{n\to\infty}\|\varphi_n\|=1$.

Setting

$$h_n=-\Delta\varphi_n+(q-\lambda)\varphi_n \tag{41}$$

and denoting by ω' some domain satisfying the condition $\omega\subset\omega'\subset\Omega$, we obtain from (41) the representation

$$4\pi\varphi_n(P)=\int_{\omega'}\frac{h_n(Q)}{r(P,Q)}\,d\omega_Q+\int_{\omega'}\frac{\lambda-q(Q)}{r(P,Q)}\,\varphi_n(Q)\,d\omega_Q+\gamma_n \tag{42}$$

thus the sequence of the harmonic functions $\{\gamma_n(P)\}$ is bounded in $\mathscr{L}_2(\omega')$. Hence it follows from well-known theorems /54/ that this sequence is compact in the sense of uniform convergence in any domain $\omega''\subset\omega'$.

Since the integral operator

$$K[h]=\int_{\omega'}\frac{h(Q)}{r(P,Q)}\,d\omega_Q$$

transforms any sequence of functions bounded in $\mathscr{L}_2(\omega')$ into a sequence of functions which is compact in the sense of uniform convergence in ω, then by a second application of representation (42) together with the proven compactness of the sequence $\{\gamma_n(P)\}$ we establish the compactness of the sequence $\{\varphi_n(P)\}$ in the sense of uniform convergence in ω''.

Furthermore, it is obvious that together with the orthonormalized sequence $\{f_n\}$ the sequence

$$\frac{1}{\sqrt{2}}\{f_{2n}(P)-f_{2n-1}(P)\},$$

and, consequently, the sequence of functions

$$u_n(P)=\frac{1}{\sqrt{2}}[\varphi_{2n}(P)-\varphi_{2n-1}(P)]\qquad(n=1,\ 2,\ \ldots), \tag{43}$$

will be characteristic and, moreover,

$$\lim_{n\to\infty}\|u_n(P)\|_{\Omega}=1.$$

The symbol $\|\ldots\|_{\Omega}$ denotes here the norm in $\mathscr{L}_2(\Omega)$. A similar meaning is given to the subscript placed next to the sign of the norm or of the scalar product for any domain ω. In order to avoid misunderstanding when simultaneously considering several metrics, we replace the subscript ω by the subscript $\mathscr{L}_2(\omega)$.

Fixing the domain ω'' by the condition $\omega\subset\omega''\subset\omega'$, we may assume, without loss of generality, that the whole sequence $\{\varphi_n(P)\}$ converges uniformly in the domain ω''. The characteristic sequence $\{u_n\}$ defined by (43) converges uniformly to zero in ω'' so that

$$\lim_{n\to\infty}u_n(P)=0\qquad(P\in\omega''). \tag{44}$$

In order to pass from the sequence $\{u_n\}$ to the required sequence $\{v_n\}$ we set

$$v_n(P) = \theta(P)\, u_n(P) \qquad (P \in \Omega), \tag{45}$$

where $\theta(P)$ is some function twice continuously differentiable in Ω satisfying the conditions

$$\begin{aligned} \theta(P) &\equiv 0 \qquad (P \in \omega), \\ \theta(P) &\equiv 1 \qquad (P \in \omega''), \\ 0 \leqslant \theta(P) &\leqslant 1 \qquad (P \in \Omega). \end{aligned}$$

Since the functions $v_n(P)$ are twice continuously differentiable in Ω, and outside the domain $\omega'' \subset \Omega$ they coincide with $u_n(P) \in \mathfrak{D}_L$, then $v_n \in \mathfrak{D}_L$ $(n = 1, 2 \ldots)$.

It follows from (45), (44), and (43) that for sufficiently large N and $m > N$, $n > N$, we shall have

$$\begin{aligned} \|v_m - v_n\|_\Omega &\geqslant \|u_m - u_n\|_\Omega - \|v_m - u_m\|_\Omega - \|u_n - v_n\|_\Omega = \\ &= \|u_m - u_n\|_\Omega - \|(\theta - 1)\, u_m\|_{\omega^*} - \|(1 - \theta)\, u_n\|_{\omega^*} > 1, \end{aligned}$$

Thus it follows that the sequence $\{v_n\}$ in $\mathscr{L}_2(\Omega)$ is not compact.

We now show that the following limit relation holds in the metric of $\mathscr{L}_2(\Omega)$

$$\lim_{n \to \infty} \{-\Delta v_n + (q - \lambda)\, v_n\} = 0.$$

To this purpose, we obtain by means of (45) that

$$\begin{aligned} \|-\Delta v_n + (q - \lambda)\, v_n\|_\Omega \leqslant \|-\Delta u_n + (q - \lambda)\, u_n\|_\Omega + \\ + 2\|\{|\nabla u_n| \cdot |\nabla \theta|\}\|_{\omega^*} + \|u_n \Delta\theta\|_{\omega^*}. \end{aligned} \tag{46}$$

From relation (44) and from the fact that $\lim_{n \to \infty} h_n = 0$ if follows that the first and third terms of the right-hand side of inequality (46) tend to zero when $n \to \infty$. We therefore have to prove that*

$$\lim_{n \to \infty} \|\nabla u_n\|_{\omega^*} = 0. \tag{47}$$

We shall prove that for any point P of the domain ω'' there exists a sphere $|PQ| = \rho$ of radius ρ (independent of P), such that

$$\lim_{n \to \infty} \int\limits_{|PQ| < \rho} |\nabla u_n(Q)|^2\, d\omega_Q = 0, \tag{48}$$

thus proving (47).

We now introduce the domain ω''' satisfying the condition $\omega'' \subset \omega''' \subset \omega'$. Let b denote half the minimum distance between the points of the boundary of ω'' and the points of the boundary of ω'''.

Applying Green's second formula to the sphere $\varkappa_\rho$ defined by $|PQ| < \rho$ $(P \in \omega'', \rho < b)$, we obtain

$$(\Delta u_n,\ u_n)_{\varkappa_\rho} = \int\limits_{|PQ| = \rho} u_n \frac{\partial u_n}{\partial \rho}\, ds - \|\nabla u_n\|^2_{\varkappa_\rho},$$

or, on introducing spherical coordinates and denoting the rectangle $(0 > \vartheta < \pi,\ 0 < \varphi < 2\pi)$ by π,

$$(\Delta u_n,\ u_n)_{\varkappa_\rho} = \int\limits_{\Pi} u_n \frac{\partial u_n}{\partial \rho}\, \rho^2 \sin\vartheta\, d\vartheta\, d\varphi - \|\nabla u_n\|^2_{\varkappa_\rho},$$

* From this point on we write $\|\nabla u\|$ instead of $\|\,|\nabla u|\,\|$.

whence, integrating over ρ from a to b $(0 < a < b)$ and changing the order of the integration, we find

$$\int_a^b (\Delta u_n,\ u_n)_{x_\rho}\, d\rho = \frac{1}{2}\int_\Pi [u_n^2\rho^2]_a^b \sin\vartheta\, d\vartheta\, d\varphi - \\ - \int_a^b d\rho \int_\Pi u_n^2 \rho \sin\vartheta\, d\vartheta\, d\varphi - \int_a^b \|\nabla u_n\|^2_{x_\rho}\, d\rho$$

and, finally,

$$\int_a^b \|\nabla u_n\|^2_{x_\rho}\, d\rho = \frac{1}{2}\int_\Pi [u_n^2\rho^2]_a^b \sin\vartheta\, d\vartheta\, d\varphi - \\ - \int_a^b (h_n,\ u_n)_{x_\rho}\, d\rho - \int_a^b ([q-\lambda]\, u_n,\ u_n)_{x_\rho}\, d\rho - \\ - \int_a^b d\rho \int_\Pi u_n^2 \rho \sin\vartheta\, d\vartheta\, d\varphi.$$

Since the sequence $u_n(P)$ converges uniformly to zero in ω''', and the sequence h_n tends to zero in the metric of $\mathscr{L}_2(\Omega)$, each of the terms in the right-hand side of the last equality tends to zero for $n \to \infty$, so that

$$\lim_{n\to\infty} \int_a^b \|\nabla u_n\|^2_{x_\rho}\, d\rho = 0,$$

leading to the limit relation (47).

From (45) the sequence $\{v_n\}$ is bounded in $\mathscr{L}_2(\Omega)$. It therefore has all the required properties and the lemma is proved.

From this proof it also follows that for each value $\lambda \in C(\tilde{L})$ there exists a normalized characteristic sequence uniformly converging to zero in any domain $\omega \subset \Omega$.

Theorem 36 /31 (5)/. If the decomposition of the operator $\tilde{L}$ is defined by relations (39), then

$$C(\tilde{L}) = C(L_0). \tag{49}$$

In particular, when the domain Ω is a part of the space $\mathfrak{S}$ bounded by the closed surface ζ, or when $\Omega = \mathfrak{S}$, we have

$$C(\tilde{L}) = C(L_0) = C(L_\Gamma). \tag{50}$$

Proof. If $\lambda \in C(L_0)$, then $\lambda \in C(\tilde{L})$, since $L_0 \subset \tilde{L}$. Conversely, if $\lambda \in C(\tilde{L})$, then according to Lemma 10, there exists a characteristic sequence of functions for the operator $\tilde{L}$, vanishing identically inside some domain $\omega \subset \Omega$ which contains the decomposition boundary Γ. This sequence is therefore also characteristic for the operator L_0.

Relation (49) is thus established. Relation (50), too, is established, since by the conditions of the second part of the theorem the operator L_ω is regular, so that $C(L_\omega) = 0$, and, consequently,

$$C(L_0) = C(L_\omega) + C(L_\Gamma) = C(L_\Gamma).$$

It also follows from Lemma 10 that as in the one-dimensional case, the continuous part of the spectrum of the operator $\tilde{L}$ defined by operation (1) in Ω does not depend on the behavior of the function $q(P)$ in any domain $\omega \subset \Omega$ (however it does depend on the boundary conditions on the boundary of the domain Ω).

Whether the type of the points of $C(\tilde{L})$ is independent of the behavior of the function $q(P)$ in the domain $\omega \subset \Omega$ has yet to be investigated for the multidimensional case.

Suppose that the self-adjoint operator $\tilde{L}$ is defined in the domain $\Omega = \Omega[\sigma, \zeta]$ by operation (1) together with separated boundary conditions, and suppose that Γ is a closed decomposition surface which includes σ, then the "decomposed" operator L_0 may be written as an orthogonal sum of two symmetric operators as follows

$$L_0 = L_{\sigma, \Gamma} \oplus L_{\Gamma}, \tag{51}$$

These operators are induced by the operator $\tilde{L}$ in $\mathscr{L}_2(\Omega[\sigma, \Gamma])$ and $\mathscr{L}_2(\Omega[\Gamma, \zeta])$ respectively.

A boundary condition which is separated on the boundary of σ will be called D-regular if for some decomposition surface Γ we have $C(L_{\sigma, \Gamma}) = 0$. This definition is permissible since from theorem 36 it follows that the set $C(L_{\sigma, \Gamma})$ is independent of the choice of Γ.

Theorem 37 /31 (5)/. Let $\tilde{L}_1$ be a self-adjoint operator generated by operation (1) in the domain $\Omega[\sigma_1, \infty]$ with a D-regular boundary condition on σ_1. If $\tilde{L}_2$ is a self-adjoint operator generated by the same operation in the domain $\Omega[\sigma_2, \infty]$, with separated boundary conditions, then since the boundary conditions at infinity for the operators $\tilde{L}_1$ and $\tilde{L}_2$ are equivalent, we have $C(\tilde{L}_1) \subset C(\tilde{L}_2)$.

Proof. Decomposing the operator $\tilde{L}_1$ on the surface Γ which contains σ_1 and σ_2, we obtain

$$L_{1,0} = L_{\sigma_1, \Gamma} \oplus L_{\Gamma, \infty}.$$

Using the relation $L_{\Gamma, \infty} \subset \tilde{L}_2$ and Theorem 36, we have further

$$C(\tilde{L}_1) = C(L_{1,0}) = C(L_{\sigma_1, \Gamma}) + C(L_{\Gamma, \infty}) = C(L_{\Gamma, \infty}) \subset C(\tilde{L}_2),$$

Q. E. D.

If in addition to the conditions of Theorem 37 we suppose that the boundary condition on σ_2 is D-regular, then $C(\tilde{L}_1) = C(\tilde{L}_2)$. This supposition defines the operator $\tilde{L}_2$.

A different treatment of this theorem, which is not so general, has been given in /81/.

18. Small and relatively small perturbations in the multidimensional case. By means of the decomposition principle established in the previous sub-section for operation (1), the following generalization of Theorem 23 to the multidimensional case is easily obtained.

Theorem 38 /31 (5)/. Let l_1 and $l_2 = l_\eta$ be two operations of the type (1) in $\mathscr{L}_2(\mathfrak{G})$, and let the function $\eta(P) = q_2(P) - q_1(P)$ tend to zero for $|OP| \to \infty$. If $\tilde{L}_1$ is some self-adjoint operator generated by the operation l_1, then the operator $\tilde{L}_2$ generated by the operation l_2 on $\mathfrak{D}_{\tilde{L}_1}$ is self-adjoint and $C(\tilde{L}_1) = C(\tilde{L}_2)$.

Proof. The operator $\tilde{L}_2$ is self-adjoint since it is a sum of the self-adjoint operator $\tilde{L}_1$ and of a bounded operator representing multiplication by $\eta(P)$ (see /72 (3)/).

Let $\delta > 0$. Let the operators $\tilde{L}_1$ and $\tilde{L}_2$ be decomposed on the sphere $|OP| = \gamma$. In the region outside this sphere $|\eta(P)| < \delta$.

Making use of Theorem 36, we obtain

$$C(\tilde{L}_1) = C(L_{1,0}) = C(L_{1,\gamma}),$$
$$C(\tilde{L}_2) = C(L_{2,0}) = C(L_{2,\gamma}).$$

Extending the operators $L_{1,\gamma}$ and $L_{2,\gamma}$ to self-adjoint operators $\tilde{L}_{1,\gamma}$ and $\tilde{L}_{2,\gamma}$ by means of the regular boundary condition $u(P)=0$ for $|OP|=\gamma$, we obtain the relations

$$C(\tilde{L}_{1,\gamma})=C(L_{1,\gamma}),\quad C(\tilde{L}_{2,\gamma})=C(L_{2,\gamma}).$$

On the other hand, from the inequality

$$\|\tilde{L}_{2,\gamma}-\tilde{L}_{1,\gamma}\|<\delta$$

and Theorem 9 it follows that for any $\lambda\in C(\tilde{L}_{1,\gamma})$ we then have

$$C(\tilde{L}_{2,\gamma})\cap(\lambda-\delta,\lambda+\delta)\neq\emptyset.$$

Thus, since $\delta>0$ is arbitrary, we find that $\lambda\in C(\tilde{L}_{2,\gamma})=C(\tilde{L}_2)$, so that $C(\tilde{L}_1)\subset C(\tilde{L}_2)$. In precisely the same way it can be shown that $C(\tilde{L}_2)\subset C(\tilde{L}_1)$. The theorem is thus proved.

Passing now to relatively small perturbations, we shall assume that in (1)

$$q(P)\geqslant\alpha>0\qquad(P\in\mathfrak{G}).\tag{52}$$

Let l_η denote the operation obtaining from (1) when $q(P)$ is replaced by $q(P)+\eta(P)$.

The following theorem, due to M.Sh. Birman, is the generalization of Theorem 24 for the multidimensional case.

Theorem 39 /13 (5)/. Suppose (52) is satisfied and

$$\lim_{|OP|\to\infty}\frac{\eta(P)}{q(P)}=0,$$

then the operator L_η is self-adjoint and $C(L_\eta)=C(L)$.

Proof. Since the function

$$q(P)+\eta(P)=q(P)\left\{1+\frac{\eta(P)}{q(P)}\right\},$$

is bounded from below, it follows from Theorem 24 that the operator L_η is self-adjoint.

Let A denote the operator generated by the unperturbed operation (1) on the manifold $\mathfrak{D}_L$ of finite functions $\mathfrak{D}_L$. Let K and K_1 denote the operator representing multiplication of the functions of $\mathfrak{D}_L$ by $\eta(P)$ and $|\eta(P)|$ respectively. By virtue of Theorem 19, it is sufficient to show that the operator K_1 is completely continuous relative to A.

Choosing the radius R of the sphere $\varkappa_R$ so that for $P\in\varkappa_R$ we have $\frac{|\eta(P)|}{q(P)}<\delta^2$ where δ is some given number, we obtain for any function $u\in\mathfrak{D}_L$

$$(K_1u,\,u)=\int_{\mathfrak{G}}|\eta(P)|\cdot|u(P)|^2\,d\omega\leqslant\int_{\varkappa_R}|\eta(P)|\cdot|u(P)|^2\,d\omega+$$

$$+\delta^2\int_{\mathfrak{G}-\varkappa_R}q(P)|u(P)|^2\,d\omega\leqslant\int_{\varkappa_R}|\eta(P)||u(P)|^2\,d\omega+$$

$$+\delta^2\int_{\mathfrak{G}}(|\nabla u|^2+q|u|^2)\,d\omega=\int_{\varkappa_R}|\eta(P)|\cdot|u(P)|^2\,d\omega+\delta^2(Au,\,u).$$

This inequality is analogous to inequality (10) of No. 8 and, as in the one-dimensional case, it is sufficient to show the compactness of the set of

functions $u \in F$ which, for some finite M, satisfy the inequality

$$\int_{\Omega} (|\nabla u|^2 + q|u|^2)\, d\omega < M^2,$$

in the metric

$$[u, u]_{K_1} = \int_{\Omega_R} |\eta(P)| \cdot |u(P)|^2 d\omega.$$

The latter statement follows directly from the well-known lemma of F. Rellich (see /54/).

19. The negative part of the spectrum and the quadratic functional. As was noted in No. 7, the boundedness or unboundedness from below of the spectrum $S(\tilde{L})$ does not depend, in the one-dimensional case, on the values of the coefficients of the differential operation beyond the neighborhood of the singular end of the interval. The first part of the present subsection is devoted to the generalization of this statement to the multidimensional case.

To prove the corresponding theorem, which may be regarded as an extension of Theorem 36 to the case where $\lambda = -\infty$, the following two lemmas are necessary.

Lemma 11 /31 (5)/. If the operator $\tilde{L}$ has a sequence of eigenvalues λ_n which converges to minus infinity, where $\{\varphi_n\}$ is the corresponding orthonormalized sequence of eigenfunctions, then for any domain $\omega \subset \Omega$ the following limit relation holds

$$\lim_{n \to \infty} \|\varphi_n\|_{\omega} = 0. \tag{53}$$

Proof. Assuming this is not so, it is possible to find a number $\delta > 0$, such that for some sequence of eigenfunctions we have $\|\varphi_{n_k}\|_{\omega} > \delta$ $(k = 1, 2, \ldots)$, or renumerating the functions of this infinite sequence

$$\|\varphi_k\|_{\omega} > \delta \qquad (k = 1, 2, \ldots). \tag{54}$$

Let us choose some domain ω' $(\omega \subset \omega' \subset \Omega)$ and construct in Ω a twice continuously differentiable function $\theta(P)$, satisfying the conditions

$$\begin{aligned} \theta(P) &\equiv 1 \qquad (P \in \omega), \\ \theta(P) &\equiv 0 \qquad (P \overline{\in} \omega'), \\ 0 \leqslant \theta(P) &\leqslant 1 \qquad (P \in \Omega) \end{aligned}$$

Setting

$$\psi_k(P) = \theta(P)\varphi_k(P),$$

we obtain

$$\begin{aligned} \int_{\omega'} l[\psi_k]\, \bar{\psi}_k\, d\omega = \lambda_k \int_{\omega'} \theta^2 |\varphi_k|^2 d\omega - \\ - \int_{\omega' - \omega} \theta\, \Delta\theta\, |\varphi_k|^2 d\omega - 2 \int_{\omega' - \omega} \langle \nabla\theta, \nabla\varphi \rangle\, \theta\bar{\varphi}\, d\omega, \end{aligned} \tag{55}$$

where

$$\langle \nabla\theta, \nabla\varphi \rangle = \frac{\partial\theta}{\partial x}\frac{\partial\varphi}{\partial x} + \frac{\partial\theta}{\partial y}\frac{\partial\varphi}{\partial y} + \frac{\partial\theta}{\partial z}\frac{\partial\varphi}{\partial z}.$$

Since the left-hand side of (55) $(\tilde{L}\psi_k, \psi_k)$ and the first two terms on the right-hand side are real, the last term on the right-hand side is also real,

and consequently may be represented in the form

$$\int_{\omega'-\omega} \langle \nabla\theta,\ \nabla\varphi\rangle\,\theta\bar{\varphi}\,d\omega + \int_{\omega'-\omega} \langle \nabla\theta,\ \nabla\bar{\varphi}\rangle\,\theta\varphi\,d\omega = \\ = \frac{1}{2}\int_{\omega'-\omega} \langle \nabla\theta^2,\ \nabla|\varphi|^2\rangle\,d\omega.$$

From Green's formula we have

$$\int_{\omega'-\omega} \langle \nabla\theta^2,\ \nabla|\varphi_k|^2\rangle\,d\omega = -\int_{\omega'-\omega} |\varphi_k|^2\Delta\theta^2\,d\omega,$$

consequently the second and third terms in (54) are uniformly bounded with respect to k, so that

$$\int_{\omega'} l[\psi_k]\,\bar{\psi}_k\,d\omega < \lambda_k\int_{\omega'} |\varphi_k|^2\,d\omega + \mu,$$

where the number μ does not depend on k.

By assumption (54) it follows from the last inequality that

$$\lim_{k\to\infty} (l[\psi_k],\ \psi_k)_{\omega'} = -\infty.$$

Contradicting the obvious inequality

$$(l[\psi_k],\ \psi_k)_{\omega'} > (\psi_k,\ \psi_k)_{\omega'}\inf_{P\in\omega'} q(P).$$

Assumption (54) is therefore false and the limit relation (53) is proved.

The sequence of functions $\{\psi_n\}$ of $\mathfrak{D}_L$ is said to be characteristic for the value $\lambda = -\infty$, if it is bounded in $\mathscr{L}_2(\Omega)$ and satisfies the limit relation $\lim_{n\to\infty} (\tilde{L}\varphi_n,\ \varphi_n) = -\infty$. Such a sequence exists if and only if the operator $\tilde{L}$ is not bounded from below.

Lemma 10 may be extended to the case where $\lambda = -\infty$, as follows.

Lemma 12 /31 (5)/. If the operator $\tilde{L}$ is not bounded from below and if $\omega\subset\Omega$, then there exists a sequence of functions $\{\psi_n(P)\}$, identically zero in the domain ω, which are characteristic for the value $\lambda = -\infty$.

Proof. If the set $C(\tilde{L})$ is not bounded from below, the existence of the required characteristic sequence $\{\psi_n\}$ follows from Lemma 10. However, if the set $C(\tilde{L})$ is bounded from below, then, since the operator $\tilde{L}$ is not bounded from below, it follows that there exists a sequence of eigenvalues of $\tilde{L}$, tending to minus infinity. Using the previous lemma and fixing the domain ω' by the condition $\omega\subset\omega'\subset\Omega$, we find the normalized sequence $\{\varphi_n\}$ of the eigenfunctions of the operator $\tilde{L}$, for which

$$\lim_{n\to\infty} (L\varphi_n,\ \varphi_n) = -\infty$$

and

$$\lim_{n\to\infty} \|\varphi_n\|_{\omega'} = 0. \tag{56}$$

We now assume that for $P\in\Omega$

$$\psi_n(P) = \theta(P)\,\varphi_n(P). \tag{57}$$

where $\theta(P)$ is some twice continuously differentiable function satisfying the conditions

$$\theta(P) \equiv 0 \qquad (P \in \omega),$$
$$\theta(P) \equiv 1 \qquad (P \overline{\in} \omega'),$$
$$0 \leqslant \theta(P) \leqslant 1 \qquad (P \in \Omega).$$

As in the proof of the previous lemma, we find for the sequence $\{\psi_n\}$ that

$$(\tilde{L}\psi_n, \psi_n) = \lambda_n \|\varphi_n\|^2_{\Omega-\omega'} + \lambda_n \|\theta\varphi_n\|_{\omega'-\omega} + \mu_n, \tag{58}$$

where the quantity μ_n is bounded for $n \to \infty$.

Now relation (56) implies that

$$\lim_{n\to\infty} \|\varphi_n\|_{\Omega-\omega'} = 1,$$

and from relation (58) it follows that the functions (57) form the required characteristic sequence.

The next theorem follows directly from this lemma.

Theorem 40 /31 (5)/. If the decomposition of the operator $\tilde{L}$ is defined by relations (39), then the operators $\tilde{L}$ and L_0 are simultaneously bounded or unbounded from below. In particular, when the domain Ω is an infinite part of the space $\mathfrak{E}$ bounded by a single closed surface ζ, or when $\Omega = \mathfrak{E}$, the operators $\tilde{L}$ and L_Γ will simultaneously be bounded or unbounded from below.

It also follows from Lemma 12 that as in the one-dimensional case, *the property of boundedness from below of the spectrum of the operator generated by operation (1), does not depend on the values of the coefficient $q(P)$ in any domain $\omega \subset \Omega$ (but now it generally depends on the boundary conditions applying to the boundary of the domain Ω).*

A D-regular boundary condition, isolated on the finite boundary σ of the domain $\Omega = \Omega[\sigma, \zeta]$, is said to be regular, if in decomposition (51) the symmetric operator $L_{\sigma\Gamma}$ is bounded from below. An example of a regular isolated boundary condition on σ is a condition of the type $\left[\frac{\partial u}{\partial n} + pu\right]_{P\in\sigma} = 0$ where $p = p(P) \geqslant 0$.

The suitability of this definition, i.e., the independence of the regularity of the boundary condition of the choice of the decomposition surface Γ, follows from Lemma 12.

If a regular boundary condition is specified on the boundary σ of the domain $\Omega[\sigma, \zeta]$, then Lemma 12 obviously still holds for any domain ω lying in $\Omega[\sigma, \zeta]$ of the form

$$\omega = \Omega[\sigma, \Gamma].$$

Therefore Lemma 12 implies the following theorem.

Theorem 40bis /31 (5)/. Let $\tilde{L}_1$ be a self-adjoint operator generated by operation (1) in the domain $\Omega[\sigma_1, \infty]$ with a regular boundary condition isolated on σ_1, and let $\tilde{L}_2$ be a self-adjoint operator generated by the same operation in the domain $\Omega[\sigma_2, \infty]$ with separated boundary conditions. The boundary conditions at infinity are the same as for $\tilde{L}_1$. If the operator $\tilde{L}_1$ is unbounded from below, then also the operator $\tilde{L}_2$ is unbounded from below.

The differential operation (1) generates the quadratic functional

$$\Phi[u] = \int_\Omega |\nabla u|^2 d\omega + \int_\Omega q|u|^2 d\omega,$$

which in the present subsection is considered only on finite functions of $\mathfrak{D}_L$.

Let us assume $\Omega = \mathfrak{E}$ and call a finite function N-finite, if it vanishes identically inside the sphere $|OP| = N$.

The following theorem generalizes the first part of Theorem 28 to the multi-dimensional case.

Theorem 41 /31 (10)/. A necessary and sufficient condition for the negative part of the spectrum of the self-adjoint operator $\tilde{L}$, defined in $\mathcal{L}_2(\mathfrak{E})$ by operation (1), not to have limit points other than zero, is that for any $\varepsilon > 0$, there exists a number $N = N_\varepsilon$ such that the quadratic functional

$$\Phi_\varepsilon[u] = \int_{\mathfrak{E}} |\nabla u|^2 \, d\omega + \int_{\mathfrak{E}} q\,|u|^2 \, d\omega + \varepsilon \int_{\mathfrak{E}} |u|^2 \, d\omega$$

is non-negative on all the N-finite twice continuously differentiable functions $u = u(P)$.

Proof. Suppose that the set $(-\infty, 0) \cap S(\tilde{L})$ does not have limit points other than zero, but that for some $\varepsilon > 0$ there is no number $N = N_\varepsilon$ with the property required by the theorem. It is then possible to construct an infinite sequence of twice continuously differentiable finite functions $\{\varphi_n\}$ with disjoint supports for which

$$([L + \varepsilon I]\varphi_n, \varphi_n) < 0 \qquad (n = 1, 2, \ldots).$$

By virtue of Theorem 13 of No. 3 it follows from this inequality that the set $(-\infty, -\varepsilon) \cap S(\tilde{L})$ is infinite. This is a contradiction, since there are no limit points other than zero in the negative part of the spectrum $S(\tilde{L})$.

Conversely, assume that for any $\varepsilon > 0$ there exists a number $N = N_\varepsilon$ with the required property. The minimal operator L generated by operation (1) will then be bounded from below by virtue of Theorem 40. Thus, on the basis of Theorem 35, the operator L is self-adjoint, so that $\tilde{L} = L$. Now if the set $(-\infty, 0) \cap S(L)$ has a limit point $\lambda\,(-\infty < \lambda < 0)$ which is different from zero, then $\lambda \in C(L)$, and from Lemma 10 of No. 17, for any $N > 0$ there exists for the value λ a characteristic sequence $\{\psi_n\}$ of twice continuously differentiable N-finite functions. For any positive $\varepsilon_0 < -\lambda$ we would have for this sequence that

$$\Phi_{\varepsilon_0}[\psi_n] = ([L + \varepsilon_0 I]\psi_n, \psi_n) < 0 \qquad (n = 1, 2, \ldots).$$

This contradicts our original assumption that there exists a number $N = N_{\varepsilon_0}$ with the property required by the theorem. The theorem is thus proved. For the quadratic functional $\Phi[u]$ the following elementary proposition, corresponding to Lemma 5 of No. 10, also holds.

Lemma 13. Let the function $v(P)$ be twice continuously differentiable inside a bounded domain Ω, and continuous up to the boundary S of this domain. If $v(P)$ is equal to zero for $P \in S$, then for any $\varepsilon > 0$ there exists a function $u(P)$, which is twice continuously differentiable in Ω and equal to zero outside some domain $\omega \subset \Omega$, such that

$$|\Phi[u] - \Phi[v]| < \varepsilon.$$

Proof. Let $\omega_\delta \subset \Omega$ be a domain with the boundary σ, each point of which is at a distance $\delta > 0$ from S. Let $\theta_\delta(P)$ denote a twice continuously differentiable function in Ω, which is equal to 1 for $P \subseteq \omega$ and vanishes near the surface S. It may be assumed* that the maximum value of the modulus of the gradient of the function $\theta_\delta(P)$ is of the order of δ^{-1}.

* See footnote on page 47.

Setting $v = \theta u$, we obtain

$$\Phi[u] - \Phi[v] =$$

$$= \int_{\Omega - \omega_\delta} (1 - \theta_\delta^2)(|\nabla u|^2 + q|u|^2)\,d\omega - \int_{\Omega - \omega_\delta} u^2 |\nabla \theta_\delta|^2\,d\omega.$$

Since the first integral on the right-hand side of this equality tends to zero for $\delta \to 0$, to prove the lemma it remains to show the same for the integral

$$\int_{\Omega - \omega_\delta} u^2 |\nabla \theta_\delta|^2\,d\omega.$$

Now the second factor in this integrand is of the order of δ^{-2}, and the first factor u^2 is, by virtue of Lagrange's mean theorem, of the order of δ^2, so that for $\delta \to 0$ this integral tends to zero. The lemma is therefore proved.

Remark. The lemma still holds if we demand in addition that $v(P)$ be orthogonal in $\mathscr{L}_2(\Omega)$ to a finite number of functions, which themselves are orthogonal in $\mathscr{L}_2(\Omega)$ to $u(P)$. This follows from Lemma 3 of No. 6, which refers to linear manifolds dense in the abstract Hilbert space.

Lemma 13 thus permits a weakening of the requirement of twice continuous differentiability under the conditions of Theorem 41.

Note also that Theorem 41 in the appropriate formulation still holds for the case of a bounded domain Ω. There the singularity of the problem is caused by the singularities of the coefficient on the boundary of the domain. It can be proved that in this case the semi-boundedness of the operator L no longer implies that it is self-adjoint. Although the condition of discreteness of the spectrum to the left of the zero, indicated in Theorem 41, is evidently necessary for any self-adjoint extension $\tilde{L}$ of the operator L, it is no longer sufficient for all such extensions. However, the condition indicated in Theorem 41 is sufficient for the discreteness to the left of the zero of the spectrum $S(\mathring{L})$ of the rigid extension $\mathring{L}$ of the operator L. To prove this we have to make use of the fact that in the case of a rigid extension we have $\mathfrak{D}_{\mathring{L}} \subset H_{\mathring{L}}$, and that any function $\varphi(P)$ of $\mathfrak{D}_{\mathring{L}}$, equal to zero in the domain $\omega \subset \Omega$, may be L-approximated by functions of $\mathfrak{D}_{\mathring{L}}$ finite in Ω and equal to zero in the domain ω. To prove the last statement we first have to L-approximate $\varphi(P)$ by a function $\psi(P) \subset \mathfrak{D}_{\mathring{L}}$ finite in Ω and then L-approximate $\psi(P)$ by a twice continuously differentiable finite function, equal to zero in ω. The latter can be achieved owing to the smallness of the function $\psi(P)$ and of the modulus of its gradient in $\mathscr{L}_2(\omega)$.

We first establish the following theorem. It will be used further in No. 47.

Theorem 41 bis. ***A necessary and sufficient condition for the negative part of the spectrum of the rigid extension $\mathring{L}$ of the operator L, defined by operation (1) in $\mathscr{L}_2(\Omega)$, not to have limit points different from zero, is that for any $\varepsilon > 0$ there exists inside Ω a domain ω, such that the quadratic functional***

$$\Phi_\varepsilon[u] = \int_\Omega |\nabla u|^2\,d\omega + \int_\Omega q|u|^2\,d\omega + \varepsilon \int_\Omega |u|^2\,d\omega$$

is non-negative for all the functions which are twice continuously differentiable and finite in Ω, and equal to zero in the domain ω.

The deformation of an arbitrary domain $\Omega \subset \mathfrak{G}$ is said to be finite if it modifies only a bounded part of the boundary domain Ω.

Let $\tilde{L}_1$ be a self-adjoint operator in $\mathscr{L}_2(\Omega)$ which is generated by operation (1) with a single singular point at infinity, and the boundary condition $\frac{\partial u}{\partial n} + \rho u = 0$ $(0 \leqslant \rho(P) \leqslant \infty)$ on the boundary of the domain Ω.

Let $\tilde{L}_2$ denote the operator obtained from $\tilde{L}_1$ in the following way. The domain undergoes a finite deformation, where the new sections of the boundary satisfy a boundary condition of the above form. The potential undergoes an arbitrary finite variation while its continuous differentiability is preserved.

From the results of No. 17 and No. 19 it follows that

$$C(\tilde{L}_1) = C(\tilde{L}_2)$$

and, in addition, the point $\lambda = -\infty$ either simultaneously belongs or does not belong to both $S(\tilde{L}_1)$ and $S(\tilde{L}_2)$.

The problem of the identity of the types of points $C(\tilde{L}_1)$ and $C(\tilde{L}_2)$, however, remains open, even when only the boundary condition on a finite section of the boundary of the domain Ω varies, or under a finite variation of the potential only.

Let us consider in more detail the problem of the type of the first point λ_0 of the continuous part of the spectrum of the operator $\tilde{L}_1$. Without loss of generality we may take $\lambda_0 = 0$. We distinguish between the case when the negative part of the spectrum is an infinite set (the case *af*) and the case when it is finite (the case *f*).

If the negative part of the spectrum of the operator $\tilde{L}_1$ extends to $-\infty$, then it follows from theorem 40 that the property *af* is invariant with respect to a finite perturbation of the potential of a finite part of the boundary and of the boundary condition on it.

In general, however, the problem of the conservation of the properties *f* and *af* under the above perturbations remains open. Let us assume, in addition, that together with $\tilde{L}_1$ the operator generated by the operation $-(1-\theta)\Delta u + q_1 u$ (or $-(1+\theta)\Delta u + q_1 u$) for some $\theta > 0$ and under the same conditions on the boundary S of the domain Ω, has the property *f* or *af* respectively. (These are the strengthened properties f and af). It can be shown /33/ that the strengthened properties *f* and *af* are invariant under finite perturbations of the potential $q(P)$, a finite part σ of the boundary S, and of the function $\rho(P) \geqslant 0$ in the boundary condition $\frac{\partial u}{\partial n} + \rho u = 0$ on σ. When the condition $u = 0$ is given on the variable part σ of the boundary S, this statement remains valid providing the zero boundary condition is maintained on σ after the deformation as well.

20. S. L. Sobolev's spaces and the decomposition principle. The decomposition principle, which was substantiated in No. 17 with the aid of Lemma 10 for the simplest elliptic differential operation (1) with three independent variables, may be extended to elliptic operations of a general form with any number of independent variables. It is then expedient to use imbedding theorems based on the investigation of these integrals instead of directly using potential-type integrals.

Let ω be a bounded domain of an m-dimensional Euclidian space $\mathcal{E}_m$ consisting of points P, with the coordinates $(x_1, x_2, \ldots, x_m)$.

To construct S. L. Sobolev's spaces (see, for example /89/) we first introduce a set W of all the functions $u(P)$, having in ω partial derivatives with respect to all the coordinates up to order r inclusive, and continuous up to the boundary S of the domain ω.

Defining the norm of any function $u \in W$ by the expression

$$\left\{ \int_\omega \left(\sum_{k=0}^{r} \sum_{s_1, s_2, \ldots, s_k = 1}^{m} \left| \frac{\partial^k u}{\partial x_{s_1} \partial x_{s_2} \ldots \partial x_{s_k}} \right|^2 \right) d\omega \right\}^{1/2}$$

and closing the set W in this norm, we obtain S. L. Sobolev's space $W_2^{(r)}(\omega)$.

For any natural r the space $W_2^{(r)}(\omega)$ is a Hilbert space. In particular, for $r = 0$ we have

$$\|u\|^2_{W_2^{(0)}(\omega)} = \int_\omega |u|^2 d\omega = \|u\|^2_{\mathcal{L}_2(\omega)},$$

and for $r = 1$ and $r = 2$ we have respectively

$$\|u\|^2_{W_2^{(1)}(\omega)} = \int_\omega \left(|u|^2 + \sum_{s=1}^{m} \left| \frac{\partial u}{\partial x_s} \right|^2 \right) d\omega = \int_\omega (|u|^2 + |\nabla u|^2)\, d\omega$$

and

$$\|u\|^2_{W_2^{(2)}(\omega)} = \int_\omega \left(|u|^2 + \sum_{s=1}^{m} \left| \frac{\partial u}{\partial x_s} \right|^2 + \sum_{s,t=1}^{m} \left| \frac{\partial^2 u}{\partial x_s \partial x_t} \right|^2 \right) d\omega.$$

From the theorems of S. L. Sobolev and V. I. Kondrashev it follows that for $r' > r$ we have

$$W_2^{(r')}(\omega) \subset W_2^{(r)}(\omega)$$

and that the inclusion operator from $W_2^{(r')}(\omega)$ in $W_2^{(r)}(\omega)$ is completely continuous /90 (20)/.

We now introduce the second-order symmetric elliptic operation of the general form

$$l[u] = \sum_{j,k=1}^{m} \frac{\partial}{\partial x_j} a_{jk} \frac{\partial u}{\partial x_k} + qu \tag{59}$$

with real coefficients $a_{jk}(P) = a_{kj}(P)$ $(j, k = 1, 2, \ldots, m)$ and $q(P)$ defined in the domain Ω of the space $\mathfrak{E}_m$ (in particular, it may be that $\Omega = \mathfrak{E}_m$). It is assumed that the coefficients $a_{jk}(P)$ are continuous in every bounded domain $\omega \subset \Omega$ and have bounded generalized first-order derivatives there /89/. We also assume that the function $q(P)$ is measureable and bounded in ω. The ellipticity of operation (59) implies that in each domain $\omega \subset \Omega$ the quadratic form

$$\sum_{j,k=1}^{m} a_{jk}(P) \xi_j \xi_k$$

has a positive lower bound $\alpha = \alpha(\omega)$ independent of the point $P \in \omega$, so that

$$\sum_{j,k=1}^{m} a_{jk}(P) \xi_j \xi_k \geqslant \alpha \sum_{j=1}^{m} \xi_j^2.$$

The minimum operator L is defined as the closure of the symmetric operator $\dot{L}$, generated in $\mathscr{D}_2(\Omega)$ by operation (59) on twice continuously differentiable finite functions. The inclusion of operation (59) in the theory of symmetric operators, and the theoretic-functional characterization of the domains of definition of the operators generated by it are given in the papers of O. A. Ladyzhenskaya /57/ and M. Sh. Birman /13 (1)/, /13 (4)/.

As was noted by M. Sh. Birman, the proof given in No. 17 of the decomposition lemma can be generalized to the case of operation (59) in the following manner.

Let $\omega \subset \Omega$. Without loss of generality we may consider the inequality

$$q(P) \geqslant q_0 > 0 \qquad (P \in \omega') \tag{60}$$

as being satisfied in some domain ω' $(\omega \subset \omega' \subset \Omega)$. This can be ensured by adding to the function $q(P)$ a sufficiently large positive constant μ, and replacing λ by $\lambda + \mu$. Let L_S denote the regular self-adjoint operator defined by operation (59) with the boundary condition

$$u(P)|_{P \in S} = 0.$$

Furthermore, let $\{f_n\}$ be some orthonormalized characteristic sequence for a given $\lambda \in C(\tilde{L})$. By virtue of the remark in No. 3 (page 32) it may be considered real. Then for some $a > 0$ we shall have for all n that

$$\|\tilde{L} f_n\| \leqslant \|\tilde{L} f_n - \lambda f_n\| + |\lambda| \|f_n\| < a + |\lambda|. \tag{61}$$

We now introduce the domain ω'' $(\omega \subset \omega'' \subset \omega')$ and denote by $\theta_1(P)$ a function which is twice continuously differentiable in the domain Ω, and satisfies the

condition

$$\theta_1(P) \equiv 1 \qquad (P \in \omega''),$$
$$\theta_1(P) \equiv 0 \qquad (P \in \bar{\omega}'),$$
$$0 \leqslant \theta_1(P) \leqslant 1 \qquad (P \in \Omega).$$

Denoting

$$L_s[\theta_1 f_n] = g_n \tag{62}$$

and using inequality (60), we obtain

$$\|f_n\|^2_{W_2^{(1)}(\omega'')} \leqslant \|\theta_1 f_n\|^2_{W_2^{(1)}(\omega')} \leqslant c^2 (L_S \theta_1 f_n, \theta_1 f_n)_{\mathcal{L}_2(\omega')} =$$
$$= c^2 \left\| L_S^{-\frac{1}{2}} g_n \right\|_{\mathcal{L}_2(\omega')}$$

or

$$\|f_n\|_{W_2^{(1)}(\omega'')} \leqslant c \left\| L_S^{-\frac{1}{2}} g_n \right\|_{\mathcal{L}_2(\omega')} \tag{63}$$

We now prove that the sequence $\{f_n\}$ is compact in the metric of $W_2^{(1)}(\omega'')$. Since the operator $L_S^{-\frac{1}{2}}$ in (63) is completely continuous, it is sufficient to show that the sequence $\{g_n\}$, defined by (62), is bounded in $\mathcal{L}_2(\omega')$.

We have

$$g_n = l[\theta_1 f_n] = \theta_1 l[f_n] - 2 \sum_{j,k=1}^{m} a_{jk} \frac{\partial f_n}{\partial x_j} \frac{\partial \theta_1}{\partial x_k} +$$
$$+ f_n \sum_{j,k=1}^{m} \frac{\partial}{\partial x_j} a_{jk} \frac{\partial \theta_1}{\partial x_k},$$

so that

$$\|g_n\| < \|l[f_n]\|_{\mathcal{L}_2(\omega')} + C \left\{ \sum_{k=1}^{m} \left\| \frac{\partial f_n}{\partial x_k} \right\|_{\mathcal{L}_2(\omega')} + \|f_n\|_{\mathcal{L}_2(\omega')} \right\} <$$
$$< \|\tilde{L} f_n\|_{\mathcal{L}_2(\Omega)} + mC \|f_n\|_{W_2^{(1)}(\omega')}. \tag{64}$$

The first term on the right-hand side of inequality (64) is bounded by virtue of (61). To prove the lemma it remains to show the boundedness of the second term, that is, the boundedness of the sequence $\{f_n(P)\}$ in the metric $W_2^{(1)}(\omega')$.

To obtain the necessary estimate of the magnitude of $\|f_n\|_{W_2^{(1)}(\omega')}$, we construct the function $\theta_2(P)$ for the domains ω'' and ω' $(\omega' \subset \omega'' \subset \Omega)$ in the same way as we constructed the function $\theta_1(P)$ for the domains ω' and ω $(\omega \subset \omega' \subset \Omega)$. We ma then consider inequality (60) as satisfied in the domain ω''.

We have for the real sequence $\{f_n\}$

$$\int_{\omega''} l[f_n] \theta_2 f_n \, d\omega = \int_{\omega''} \theta_2 \left\{ \sum_{j,k=1}^{m} a_{jk} \frac{\partial f_n}{\partial x_j} \frac{\partial f_n}{\partial x_k} + q f_n^2 \right\} d\omega +$$
$$+ \int_{\omega''} f_n \sum_{j,k=1}^{m} a_{jk} \frac{\partial f_n}{\partial x_j} \frac{\partial \theta_2}{\partial x_k} d\omega,$$

but

$$\int_{\omega''} f_n \sum_{j,k=1}^{m} a_{jk} \frac{\partial f_n}{\partial x_j} \frac{\partial \theta_2}{\partial x_k} d\omega = \frac{1}{2} \int_{\omega''} \sum_{j,k=1}^{m} a_{jk} \frac{\partial f_n^2}{\partial x_j} \frac{\partial \theta_2}{\partial x_k} d\omega =$$
$$= -\frac{1}{2} \int_{\omega''} f_n^2 \sum_{j,k=1}^{m} \frac{\partial}{\partial x_j} a_{jk} \frac{\partial \theta_2}{\partial x_k} d\omega,$$

so that

$$\int_{\omega''} \theta_2 \left\{ \sum_{j,k=1}^{m} a_{jk} \frac{\partial f_n}{\partial x_j} \frac{\partial f_n}{\partial x_k} + q f_n^2 \right\} d\omega =$$

$$= \int_{\omega''} l[f_n] \theta_2 f_n \, d\omega + \frac{1}{2} \int_{\omega''} f_n^2 \sum_{j,k=1}^{m} \frac{\partial}{\partial x_j} a_{jk} \frac{\partial \theta_2}{\partial x_k} d\omega. \tag{65}$$

On the other hand

$$\|f_n\|^2_{W_2^{(1)}(\omega')} \leqslant C^2 \int_{\omega'} \left\{ \sum_{j,k=1}^{m} a_{jk} \frac{\partial f_n}{\partial x_j} \frac{\partial f_n}{\partial x_k} + q f_n^2 \right\} d\omega <$$

$$\leqslant C^2 \int_{\omega''} \theta_2 \left\{ \sum_{j,k=1}^{m} a_{jk} \frac{\partial f_n}{\partial x_j} \frac{\partial f_n}{\partial x_k} + q f_n^2 \right\} d\omega. \tag{66}$$

From (65) and (66) follows the inequality

$$\|f_n\|^2_{W_2^{(1)}(\omega')} \leqslant C^2 (l[f_n], \theta_2 f_n)_{\mathfrak{L}_2(\omega'')} + C_1 \|f_n\|^2_{\mathfrak{L}_2(\omega'')} \leqslant$$

$$\leqslant C^2 \left\{ \|\tilde{L} f_n\|_{\mathfrak{L}_2(\Omega)} + \|f_n\|_{\mathfrak{L}_2(\Omega)} \right\} \cdot \|f_n\|_{\mathfrak{L}_2(\Omega)},$$

thus the sequence $\{f_n\}$ is bounded in $W_2^{(1)}(\omega')$.

The compactness of the characteristic sequence $\{f_n\}$ in $W_2^{(1)}(\omega'')$ is thus proved.

Without loss of generality, we shall consider the whole characteristic sequence $\{f_n\}$ to be convergent in $W_2^{(1)}(\omega'')$. Then the sequence of functions $\{u_n\}$ defined by

$$u_n(P) = \frac{1}{\sqrt{2}} \{f_{2n}(P) - f_{2n-1}(P)\},$$

is obviously characteristic for the value $\lambda \in C(\tilde{L})$, and in the metric of $W_2^{(1)}(\omega'')$ it converges to zero.

To construct the required sequence $\{v_n\}$ we assume, as in No. 17,

$$v_n(P) = \theta(P) u_n(P) \qquad (P \in \Omega), \tag{67}$$

where $\theta(P)$ is some function twice continuously differentiable in Ω, and satisfying the conditions

$$\begin{aligned} \theta(P) &\equiv 0 \qquad (P \in \omega), \\ \theta(P) &\equiv 1 \qquad (P \in \omega''), \\ 0 \leqslant \theta(P) &\leqslant 1 \qquad (P \in \Omega''). \end{aligned}$$

We have to show that the sequence $\{v_n\}$ possesses all the required properties.

Obviously

$$v_n(P) \equiv 0 \qquad (P \in \omega;\ n = 1, 2, \ldots).$$

It remains to be shown that the sequence $\{v_n\}$ is characteristic for the value $\lambda \in C(\tilde{L})$.

It follows directly from (67) that the sequence $\{v_n\}$ is bounded in $\mathfrak{L}_2(\Omega)$. Its non-compactness in $\mathfrak{L}_2(\Omega)$ is shown as in No. 17.

Finally

$$l[v_n] - \lambda v_n = \theta (l[u_n] - \lambda u_n) - 2 \sum_{j,k=1}^{m} a_{jk} \frac{\partial \theta}{\partial x_j} \frac{\partial u_n}{\partial x_k} - u_n \sum_{j,k=1}^{m} \frac{\partial}{\partial x_j} a_{jk} \frac{\partial \theta}{\partial x_k},$$

and consequently,

$$\|\tilde{L}v_n - \lambda v_n\|_{\mathfrak{L}_2(\Omega)} \leqslant$$

$$\leqslant \|Lu_n - \lambda u_n\|_{\mathfrak{L}_2(\Omega)} + A\sum_{k=1}^{m}\left\|\frac{\partial u}{\partial x_k}\right\|_{\mathfrak{L}_2(\omega'')} + B\|u_n\|_{\mathfrak{L}_2(\omega'')},$$

where A and B are some positive constants.

Since the sequence $\{u_n\}$ converges to zero in the metric $W_2^{(1)}(\omega'')$, the following limit relation results from the above inequality

$$\lim_{n\to\infty}\|\tilde{L}v_n - \lambda v_n\|_{\mathfrak{L}_2(\Omega)} = 0.$$

By the same token the decomposition lemma is proved for operation (59).

§ 4. J-self-adjoint operators

21. The domain of values of a quadratic functional and the external regularity field of a linear operator. If the operator A is symmetric, then the set of values of the quadratic functional $\Phi[f] = (Af, f)$ for $\|f\| = 1$ $(f \in \mathfrak{D}_A)$ lies on the real axis. The set is obviously connected since together with the values $\Phi[f_1] = \lambda_1$ and $\Phi[f_2] = \lambda_2$ $(\|f_1\| = \|f_2\| = 1)$ the real continuous function

$$F(t) = \Phi\left[\frac{f}{\|f\|}\right],$$

where $f = (1-t)f_1 + tf_2$, takes on all the intermediate values between λ_1 and λ_2 when $0 \leqslant t \leqslant 1$.

Let $K(A)$ denote the closure of the set of values of the quadratic functional (Af, f) for $\|f\| = 1$ $(f \in \mathfrak{D}_A)$, and let $R(A)$ denote its complement to the whole λ-plane.

If $K(A)$ covers the whole λ-axis, then $R(A)$ consists of the upper and lower λ-half-planes which always belong to the regularity field $\Pi(A)$ of the symmetric operator A. If the set $K(A)$ does not cover the whole λ-plane then it is a segment or a semi-axis. The operator A is then bounded or semi-bounded, and the part of the real λ-axis which does not contain points of $K(A)$ also belongs to $\Pi(A)$.

Thus, in the case of a symmetric operator A, we always have

$$R(A) \subset \Pi(A). \tag{1}$$

Bearing in mind the generalization of (1) to the case of any closed operators T, we first consider the case of dissipative operators.

A closed operator T with a domain of definition $\mathfrak{D}_T$ dense in H is called dissipative, if

$$\operatorname{Im}(Tf, f) \geqslant 0$$

for all $f \in \mathfrak{D}_T$.

If the operator T is dissipative, then for any $f \in \mathfrak{D}_T$ and $\operatorname{Im}\lambda < 0$, we have from the equality

$$Tf - \lambda f = g$$

or

$$(Tf, f) - \lambda(f, f) = (g, f)$$

that

$$|\operatorname{Im}\lambda| \cdot (f, f) \leqslant |\operatorname{Im}(g, f)| \leqslant |(g, f)| \leqslant \|g\| \cdot \|f\|.$$

hence

$$\|f\| \leqslant \frac{1}{|\operatorname{Im}\lambda|}\|g\|. \tag{2}$$

From inequality (2) it follows that the lower λ-half-plane belongs to the regularity field of the dissipative operator T.

Passing to arbitrary closed linear operators T, we denote by $K(T)$ the closure of the domain of values of the quadratic functional (Tf, f) for $\|f\| = 1$ $(f \in \mathfrak{D}_T)$

Theorem .42. For any closed linear operator T in H the totality of values of the quadratic functional (Tf, f) for $\|f\|=1$ $(f \in \mathfrak{D}_T)$ is some convex set of the complex λ-plane. If the point λ does not belong to the closure $K(T)$ of this set, then it is a regular-type point of the operator T and

$$\|(T - \lambda I) f\| \geqslant \delta \|f\|. \tag{3}$$

where δ is the distance of the point λ from the set $K(T)$.

Proof of the first statement (see /94) follows from Hausdorff's well-known theorem /104/, according to which the set $K(A)$ is convex in the finite dimensional case.

Suppose the points $\lambda_1 = (Tf_1, f_1)$ and $\lambda_2 = (Tf_2, f_2)$ $(\|f_1\| = \|f_2\| = 1)$ belong to $K(T)$, and λ is any point of the segment connecting the points λ_1 and λ_2.

Let P denote the projection operator onto the linear envelope $\mathfrak{E}$ of the vectors f_1 and f_2. We have

$$(Tf_1, f_1) = (Tf_1, Pf_1) = (PTf_1, f_1),$$
$$(Tf_2, f_2) = (Tf_2, Pf_2) = (PTf_2, f_2),$$

and, according to Hausdorff's theorem for the two-dimensional case (see /107/), there exists a normalized vector $f \in \mathfrak{E}$, for which $(PTf, f) = \lambda$.

Thus,

$$\lambda = (PTf, f) = (Tf, Pf) = (Tf, f)$$

and the convexity of the set $K(T)$ in the general case is proved.

Suppose $\lambda_0 \notin K(T)$, and w_0 is a point of the set $K(T)$, in which the distance between the point λ_0 and $K(T)$ is obtained. Let $\mathfrak{G}$ denote the straight line passing through w_0 and perpendicular to the segment connecting the points λ_0 and w_0. $\mathfrak{G}$ will obviously be the line of support of the convex set $K(T)$ at the point w_0.

If $\mathfrak{G}$ coincides with the real λ-axis, then the operator T is dissipative and (3) follows from (2). In the general case, we construct the linear transformation

$$\mu = a\lambda + b \qquad (|a| = 1),$$

which transforms the line $\mathfrak{G}$ into the line $\operatorname{Im}\mu = 0$, so that the half-plane in which the set $K(T)$ lies is transformed into the half-plane $\operatorname{Im}\mu > 0$.

The operator $S = aT + bI$ is obviously dissipative, and since the point $\mu_0 = a\lambda_0 + b$ lies in the lower μ-half-plane, then, according to (2), we have

$$\|Sf - \mu_0 f\| \geqslant |\operatorname{Im}\mu_0| \cdot \|f\| = \delta\|f\|.$$

but $S-\mu_0 J=T-\lambda_0 J$, thus the theorem is proved.

The part $R(T)$ of the regularity field $\Pi(T)$ is said to be the external regularity field. Since it is the complement of the convex set to the whole plane, the external regularity field $R(T)$ is at most doubly connected.

If the external regularity field is simply connected, then the defect number $m(\lambda)$ of the manifold $(T-\lambda I)\mathfrak{D}_T$ is the same at all the points $\lambda\in R(T)$, and when $R(T)$ is doubly connected $m(\lambda)$ is some constant in each of the doubly connected components of the domain $R(T)$.

If $\lambda\in R(T)$ the number $m(\lambda)$ is called the external defect number of the linear operator T. Thus each closed linear operator T for which the set $R(T)$ is not empty has one or two external defect numbers which may be equal.

Inside $K(T)$ there may also be regular-type points of the operator T, but in them the values of $m(\lambda)$ may differ from the external defect numbers.

To conclude thepresent subsection we recall that by the characteristic of a closed linear operator T (see /36/) at a given point λ, we mean the pair of numbers $\{n(\lambda), m(\lambda)\}$, where $n(\lambda)$ is the dimensionality of the zero subspace of the operator $T-\lambda I$. If both numbers $n(\lambda)$ and $m(\lambda)$ are finite, then the characteristic of the operator T at the point λ is called finite.

22. J-symmetric operators and their J-self-adjoint extensions. A linear operator A with a domain of definition $\mathfrak{D}_A$ dense in H is said to be J-symmetric, if for some conjugation operator of J in H

$$JAJ\subset A^*; \tag{4}$$

in particular, if

$$JAJ=A^*,$$

then the operator A is said to be J-self-adjoint. It follows from (4) that any J-symmetric operator admits closure. Therefore, in what follows any abstractly given J-symmetric operator is always considered to be closed.

For J-symmetric operators (as for symmetric operators) the following inequality holds for any λ,

$$n(\lambda)\leqslant m(\lambda), \tag{5}$$

where the numbers $n(\lambda)$ and $m(\lambda)$ are defined as in the previous subsection.

If the operator A is J-symmetric and if $(A-\lambda I)h=0$, then $JAJ\cdot Jh-\overline{\lambda}Jh=0$. Thus, by virtue of (4), we have $(A^*-\overline{\lambda}I)Jh=0$, so that $Jh\perp(A-\lambda I)\mathfrak{D}_A$, and consequently, $n(\lambda)\leqslant m(\lambda)$.

We therefore have that for J-self-adjoint operators the following relation holds for any λ

$$n(\lambda)=m(\lambda). \tag{6}$$

Thus if follows that J-self-adjoint operators (as well as self-adjoint operators) do not have a residual spectrum.

If A is a J-self-adjoint operator, then at any point λ of its regularity field $m(\lambda)=0$. Conversely, if $m(\lambda)=0$ at some point λ of the regularity field of a J-symmetric operator A, then this operator is J-self-adjoint.

In fact, if $A=JA^*J$, but for some $\lambda\in\Pi(A)$ we have $m(\lambda)\neq 0$, then by virtue of (6) we also have $n(\lambda)\neq 0$, which contradicts our assumption that $\lambda\in\Pi(A)$.

Conversely, suppose $m(\lambda)=0$ for some $\lambda\in\Pi(A)$ so that

$$(A-\lambda I)\mathfrak{D}_A=H,$$

and therefore

$$(JAJ-\bar{\lambda}I)J\mathfrak{D}_A=H. \tag{7}$$

If $JAJ\neq A^*$, then there exists an element $f\in\mathfrak{D}_{A^*}$ not belonging to $\mathfrak{D}_{JAJ}$. Setting

$$(A^*-\bar{\lambda}I)f=h, \tag{8}$$

we note, on the other hand, that by virtue of (7), there exists an element $g\in\mathfrak{D}_{JAJ}$ such that $(JAJ-\bar{\lambda}J)g=h$, or, by virtue of (4),

$$(A^*-\bar{\lambda}I)g=h. \tag{9}$$

From (8) and (9) it follows that for $\psi=g-f$

$$A^*\psi-\bar{\lambda}\psi=0. \tag{10}$$

Since $m(\lambda)=0$, equality (10) only holds for $\psi=0$; hence we obtain that $g=f$, but this is impossible, since $g\in\mathfrak{D}_{JAJ}$, and $f\notin\mathfrak{D}_{JAJ}$. Therefore we must have $JAJ=A^*$, Q. E. D.

As a corollary to the above proposition we have that if $m(\lambda_0)=0$ at some regular-type point λ_0 of a J-symmetric operator A, then $m(\lambda)=0$ at any point λ of the regularity field. The regularity field may also not be connected.

By using formulas analogous to those of Neumann-Vishik /24 (1)/ for the domain of definition of a self-adjoint operator, it can be shown /43/ that at all the points λ of the regularity field of a J-symmetric operator A, the number $m(\lambda)$ has the same value which in what follows will be called the d e f e c t n u m b e r of the operator A, and will be denoted by Def A. It can also be shown /31 (8)/ that if the external regularity field of the operator A (or, more generally, its regularity field $\Pi(A)$ /43/) is not empty, it admits J-self-adjoint extensions which preserve the belonging of a regularly given point $\lambda\in\Pi(A)$ to the field. It has been established /26/ that there exists a J-self-adjoint extension of any J-symmetric operator.

23. Some spectral properties of J-self-adjoint extensions of J-symmetric operators with a finite defect number. The properties given below of the spectrum of J-self-adjoint extensions $\tilde{A}$ of a J-symmetric operator A with a finite defect number, follow directly from general theorems of the theory of linear equations in normalized spaces which were established in /77/ and developed in a number of other papers (see /36/).

We bring, without proof, the formulation of two of these general theorems as well as other necessary information from the theory of linear equations in normalized spaces. This information, and proof of the subsequent theorems 41 and 42, can be easily found in the above mentioned fundamental paper of M. G. Krein and I. Ts. Gokhberg (/36/, Theorems 3.3 and 4.2).

Let T be an arbitrary closed linear operator in H, and $\Phi(T)$ the set of all points of the λ-plane in which the characteristic of the operator $T-\lambda I$ is finite, and the manifold $(T-\lambda I)\mathfrak{D}_T$ is closed. The set $\Phi(T)$ is open and, consequently, consists of a finite or denumerable number of connected components.

Theorem 43 /36/. If T is a closed linear operator and G some connected component of the domain $\Phi(T)$, then at all the points $\lambda \in G$, with the possible exception of some isolated points, $n(\lambda)$ will have a constant value $n(\lambda) = n$, and for the above-mentioned isolated points we have

$$n(\lambda) > n.$$

The following theorem shows that although the multiplicity of an eigenvalue never exceeds its rank, under certain conditions, because the multiplicity of an eigenvalue is finite, it follows that its rank is finite too.

Theorem 44 /36/. If $\lambda_0 \in \Phi(T)$ is an isolated eigenvalue of a closed linear operator T in H, then the rank r of the eigenvalue λ_0 is finite, and the corresponding root subspace is separable (see No. 1). Since the rank r is finite, it then follows that the representation of the H space in the form indicated in No. 1 of Chapter I is single-valued.

Theorems 43 and 44 lead to the following theorem regarding the character of the spectrum of the J-self-adjoint extensions of a J-symmetric operator A with a finite defect number in its external regularity field $R(A)$.

Theorem 45. If A is a J-symmetric operator with a finite defect number, and $\tilde{A}$ is any J-self-adjoint extension of it, then:

1. In the domain $R(A)$ there exist no points of the spectrum of the operator $\tilde{A}$ other than its eigenvalues.

2. Each point of at least one of the connected components of the domain $R(A)$ is either an eigenvalue of the operator $\tilde{A}$, or the spectrum of the operator $\tilde{A}$ does not have finite limit points inside $R(A)$.

3. In the latter case, if the set of eigenvalues of the operator $\tilde{A}$ lying in the given connected component of the domain $R(A)$ is isolated, then the rank of each eigenvalue λ_0 of this component is finite, and the corresponding root subspace O_{λ_0} is separable, according to the formula

$$H = O_{\lambda_0} \dotplus F_{\lambda_0}, \tag{11}$$

where

$$F_{\lambda_0} = H \ominus JO_{\lambda_0}. \tag{12}$$

Proof. Since the extension $\tilde{A}$ of the operator A is finite dimensional, then

$$C(\tilde{A}) \cap R(A) = C(A) \cap R(A) = 0. \tag{13}$$

Furthermore the fact that the operator $\tilde{A}$ is J-self-adjoint implies that it has no residual spectrum. Thus statement 1 is proved.

Further, it follows from (13) that the manifold $(\tilde{A} - \lambda I)\mathfrak{D}_\chi$ is closed for any $\lambda \in R(A)$. In addition, since the finite dimensionality of the extension $\tilde{A}$ implies that $n(\lambda)$ is finite, then

$$R(A) \subset \Phi(\tilde{A}). \tag{14}$$

Now, if there exists in some component of the domain $R(A)$ a point $\lambda_0 \in S(\tilde{A})$ such that $n(\lambda_0) = 0$, then, by virtue of Theorem 43, at all points of the considered connected component, with the possible exception of some isolated points, the equality $n(\lambda_0) = 0$ holds.

Statement 2 is thus proved.

Further, by virtue of (14), statement 3 follows directly from Theorem 44 without making use of statement (12).

Passing to the proof of the method of isolation of the root subspace indicated in point 3 of the theorem, we first note that the subspace F_{λ_0}, defined by relationship (12), is invariant relative to the operator $\tilde{A}$. In fact, if $f \in F_{\lambda_0} \cap \mathfrak{D}_{\tilde{A}}$, then $f \perp JG_{\lambda_0}$, and since for any $g \in G_{\lambda_0}$ we have $\tilde{A}g \in G_{\lambda_0}$, then $(\tilde{A}f, Jg) = (f, J\tilde{A}g) = 0$, that is $\tilde{A}f \in F_{\lambda_0}$.

It now remains to prove that the subspaces G_{λ_0} and $H \ominus JG_{\lambda_0}$ are linearly independent, that their sum gives the whole H space, and that in these circumstances the operator $\tilde{A} - \lambda_0 I$ is continuously transformable into F_{λ_0}.

The proof of the independence of the subspaces G_{λ_0} and $H \ominus JG_{\lambda_0}$ is not difficult, providing we use the appropriate considerations from linear algebra (see /65/).

From the linear independence of the subspaces G_{λ_0} and $H \ominus JG_{\lambda_0}$ it follows that their direct sum coincides with H. In fact, if this were not so there would exist an element $h \neq 0$, orthogonal to G_{λ_0} and $H \ominus JG_{\lambda_0}$, that is, λ belongs to the subspaces $H \ominus JG_{\lambda_0}$ and JG_{λ_0}. This is impossible, since the subspaces $J(H \ominus JG_{\lambda_0})$ and JG_{λ_0} are linearly independent together with G_{λ_0} and $H \ominus JG_{\lambda_0}$.

From the linear independence of the subspaces G_{λ_0} and $H \ominus JG_{\lambda_0}$ it also follows that the operator $\tilde{A} - \lambda_0 I$ is continuously convertible into $F_{\lambda_0} = H \ominus JG_{\lambda_0}$. Assuming the opposite, there would exist in F_{λ_0} an eigenvector of the operator $\tilde{A}$ corresponding to the value λ_0. But such an eigenvector should belong to G_{λ_0}, which again contradicts the linear independence of the subspaces G_{λ_0} and $H \ominus JG_{\lambda_0}$.

The single-valued representation of any element $h \in H$ in the form

$$h = g + f \qquad (g \in G_{\lambda_0},\ f \in H \ominus JG_{\lambda_0})$$

defines the non-orthogonal projection operators P and Q by the formulas

$$Ph = g, \quad Qh = (I - P)h = f.$$

Since $Ph \in \mathfrak{D}_{\tilde{A}}$ always, then from $h \in \mathfrak{D}_{\tilde{A}}$ it follows that $Qh \in \mathfrak{D}_{\tilde{A}}$.

Statement 3 is now established and the theorem is completely proved.

24. J-symmetric differential operators. The one-dimensional J-symmetric differential operation has the same form as the symmetric operation (1) of No. 7, but the coefficients $p_k(x)$ $(k = 0, 1, \ldots, n)$ are no longer assumed real. The simplest multi-dimensional J-symmetric operation has the form (1) of No. 15, where the potential $q(P)$ is assumed complex-valued. From here onwards the operation J in the functional space means the complex conjugate, so that $Jf(P) = \overline{f(P)}$.

The construction of the minimal J-symmetric operator L is carried out similarly to the symmetric case.

Let us denote by $\tilde{L}$ some J-self-adjoint extension of the operator L.

Let us confine ourselves in the present subsection to the simplest J-symmetric differential operations

$$-y'' + q(x)y \qquad (-\infty < x < \infty) \tag{15}$$

and

$$-\Delta u + q(P)u \qquad (P \in \mathfrak{E}). \tag{16}$$

We shall make use of the following criterion for J-self-adjointness, which in the case of a real potential corresponds to Theorem 34.

Theorem 46. If the real part of the potential in (15) or (16) is bounded from below, then the corresponding minimal operator L is J-self-adjoint.

Proof. Without loss of generality, we take the real part of the potential to be positive.

In the one-dimensional case it is sufficient for the proof of the theorem to show that for any pair of functions y and z of $\mathfrak{D}_{L^*}$ we have

$$\lim_{x\to\pm\infty}\{y'z - yz'\} = 0.$$

For any function $y\in\mathfrak{D}_{L^*}$ we have

$$\int_0^N(-y''+\bar{q}y)\,\bar{y}\,dx = -y'\bar{y}\Big|_0^N + \int_0^N |y'|^2\,dx + \int_0^N \bar{q}|y|^2\,dx$$

or passing to the real parts,

$$\operatorname{Re}\int_0^N(-y''+\bar{q}y)\,\bar{y}\,dx = -\operatorname{Re}\{y'(N)\,\overline{y(N)}\} + \\ + \operatorname{Re}\{y'(0)\,\overline{y(0)}\} + \int_0^N |y'|^2\,dx + \int_0^N |y|^2 \operatorname{Re} q(x)\,dx.$$

Thus the quantity

$$-\operatorname{Re}\{y'(N)\,\overline{y(N)}\} + \int_0^N |y'|^2\,dx + \int_0^N |y|^2 \operatorname{Re} q(x)\,dx. \tag{17}$$

converges to a finite limit for $N\to\infty$.

On the other hand, since

$$\int_0^\infty |y|^2\,dx < \infty,$$

the positive function $|y|^2 = y\bar{y}$ cannot increase monotonically, therefore, for some sequence of points $N_k\to\infty$ we shall have $[y\bar{y}' + y'\bar{y}]_{x=N_k} \leqslant 0$, that is

$$\operatorname{Re}\{y'(N_k)\,\overline{y(N_k)}\} \leqslant 0. \tag{18}$$

Now, since the function $\operatorname{Re} q(x)$ is non-negative, and from inequality (18) as well as from the convergence of (17) for $N = N_k\to\infty$, it follows that the integrals

$$\int_0^\infty |y'|^2\,dx, \quad \int_0^\infty |y|^2 \operatorname{Re} q(x)\,dx.$$

converge. From the obtained relation $y'\in\mathfrak{L}_2(0,\infty)$, it follows that for any pair of functions y and z of $\mathfrak{D}_{L^*}$, the products $y'z$ and yz' belong to $\mathfrak{L}_1(0,\infty)$, and, consequently,

$$(yz' - y'z)\in\mathfrak{L}_1(0,\infty). \tag{19}$$

On the other hand, from the equality

$$\int_0^N(-y''+\bar{q}y)\,z\,dx = [-y'z + yz']_0^N + \int_0^N(-z''+\bar{q}z)\,y\,dx$$

follows the existence of the limit

$$\lim_{x\to+\infty}\{yz' - y'z\},$$

which by virtue of (19) should be equal to zero. Thus, the theorem is proved for the one-dimensional case.

In the multidimensional case, the proof is similar to the proofs of Lemma 9 and Theorem 34 given in No. 16. Instead of equality (17) of No. 16 the equality

$$\operatorname{Re}\int\limits_{|OP|=R}\bar{u}\frac{\partial u}{\partial R}\,ds = \int\limits_{|OP|<R}|\nabla u|^2\,d\omega + \int\limits_{|OP|<R}\operatorname{Re}\{\bar{u}\,\Delta u\}\,d\omega, \tag{20}$$

following from Green's formula

$$\int\limits_{|OP|=R}\bar{u}\frac{\partial u}{\partial R}\,ds = \int\limits_{|OP|<R}|\nabla u|^2\,d\Delta + \int\limits_{|OP|<R}\bar{u}\,\Delta u\,d\omega$$

should then be used. Equality (20) of No. 16 then preserves its form if the left-hand side of (20) is assumed equal to $F(R)$. Repeating the considerations made in the proof of Lemma 9 of No. 16, we obtain that together with u and $l[u]$, $|\nabla u|$ belongs to the space $\mathfrak{L}_2(\mathfrak{G})$. Further, since for complex-valued functions of real variables the inequality $\left|\frac{\partial u}{\partial R}\right| \leqslant |\nabla u|$ still holds, then repeating the previously given proof of Theorem 34 of No. 16 without modifications, we establish the equality $JLJ = L^*$, and Theorem 46 is proved.

As to the application of the decomposition method, no complications arise in the one-dimensional case, since the defect number m is finite (it satisfies the same inequalities as in the symmetric case /31 (8)/. However, the principle of decomposition which was established in No. 17 for the case of a real potential, needs to be proved again for the multidimensional operation (16).

Analysis of the proof of Lemma 10 of No. 17, from which the principle of decomposition was derived, shows that the fact that the potential is real was essentially used only in the beginning of the proof of the Lemma. There it was shown that the normalized characteristic sequence of functions $\{f_n\}$, corresponding to the value $\lambda \in C(\tilde{L})$, may be considered as orthogonal. The orthogonality of this sequence was used only once, when it was stated that the bounded sequence

$$\{f_{2n} - f_{2n-1}\}$$

is non-compact in $\mathfrak{L}_2(\Omega)$.

In the case of a complex potential, the normalized characteristic sequence corresponding to the number $\lambda \in C(\tilde{L})$ can no longer be considered orthonormalized. However, on the basis of Lemma 2 of No. 5 it is possible to isolate from the non-compact normalized sequence $\{f_n\}$ in $\mathfrak{L}_2(\Omega)$ a subsequence $\{f_{n_k}\}$, such that the sequence of differences $\{f_{n_{k+1}} - f_{n_k}\}$ is also non-compact. This fact re-establishes the validity of the principle of decomposition in the case of operation (16) with a complex valued potential.

Chapter II

THE SPECTRUM OF ONE-DIMENSIONAL TWO-TERM DIFFERENTIAL OPERATORS

25. Two-term differential operations. This term refers to operations of the form

$$(-1)^n y^{(2n)} + q(x)\, y. \tag{1}$$

where the function $q(x)$ is assumed real and absolutely integrable in any finite interval.

The operator generated by the two-term operation (1) is not only the simplest, but also the most important object for the application of the direct methods of qualitative spectral analysis, since for $n = 1$ it coincides with the one-dimensional Schroedinger operator. Because of this relation between operation (1) and the Schroedinger differential equation, the function $q(x)$ is often called the p o t e n t i a l.

Some of the theorems given below are direct generalizations of well-known results regarding the spectrum in the case of the second-order operation

$$-y'' + q(x)\, y.$$

However, in a number of cases, the general propositions of the present chapter are new with respect to the analytically obtained results, even for the particular case $n = 1$.

It should be noted that as long as there are no data on the behavior of the potential $q(x)$, it is impossible to make any statement concerning the character of the spectrum of the operator $\tilde{L}$ for finite λ even in the simplest case $n = 1$. I. M. Gel'fand and B. M. Levitan established /28/ that for any finite interval of the λ-axis and an arbitrary closed set of points of this interval, there exists a potential $q(x)$ and boundary conditions such that the spectrum of the corresponding operator $\tilde{L}$ in this interval coincides with the given set. This statement follows from the sufficient conditions obtained by them for the solvability of the inverse problem of spectral analysis for operation (1) when $n = 1$. It may be noted here that the inverse problems of qualitative spectral analysis, when the character of the behavior of the potential has to be established from data on the properties of the spectrum, have almost not been considered. The propositions of the present chapter concerning the conditions necessary for some arrangement of the parts of the spectrum are always associated with an additional assumption on the semi-boundedness from below of the coefficient $q(x)$.

Any property of the spectrum $S(\tilde{L})$ which is independent of the choice of the extension $\tilde{L}$ of the operator L is said to be h-s t a b l e, if it is conserved when the potential $q(x)$ is replaced by the potential $\frac{1}{h} q(x)$, where h is any

positive number. This alteration of the potential is obviously equivalent to passing from operation (1) to the operation

$$(-1)^n hy^{(2n)} + q(x)y. \tag{2}$$

For certain properties of the spectrum only the behavior of the negative part of the potential is essential. In these cases it is denoted by $q^*(x)$, so that

$$q^*(x) = \begin{cases} q(x) & \text{for } q(x) < 0, \\ 0 & \text{for } q(x) \geqslant 0. \end{cases}$$

26. Some auxiliary inequalities. We now give some elementary inequalities together with their proofs; these will also be used in the next chapters. As before, we denote by $\dot{C}_n(\alpha, \beta)$ the class of functions which are n-fold continuously differentiable and equal to zero in the neighborhood of the ends of the interval (α, β).

1. For any real function $y(x) \in \dot{C}_n(-\infty, \infty)$ and for any $m \leqslant n$ the following relation holds

$$\|y^{(m)}\| \leqslant \|y\|^{\frac{n-m}{n}} \|y^{(n)}\|^{\frac{m}{n}}. \tag{3}$$

This inequality holds for any functions $C_{n-1}(0, \infty)$ (not necessarily finite) which together with their n-th order derivatives belong to the space $\mathcal{L}_2(0, \infty)$ and which satisfy the conditions

$$y(0) = y'(0) = \ldots = y^{(n)}(0) = 0. \tag{4}$$

If we omit condition (4), then in the case of a semi-axis (3) must be replaced by the inequality

$$\|y^{(m)}\| \leqslant C_{mn} \|y\|^{\frac{n-m}{n}} \|y^{(n)}\|^{\frac{m}{n}}. \tag{5}$$

2. For any real function $y(x) \in C_n(-\infty, \infty)$ there exists a constant C_n independent of $y(x)$, such that

$$\|y'\|^2 \leqslant C_n (\|y^{(n)}\|^2 + \|y\|^2). \tag{6}$$

3. If the real function $y(x) \in \dot{C}_n(-\infty, \infty)$ satisfies the condition

$$\int_{-\infty}^{\infty} y^2(x)\, dx \leqslant 1$$

and $M = \max\limits_{-\infty < x < \infty} |y(x)|$, then

$$\|y^{(n)}\| > \frac{M^{2n}}{2^n}. \tag{7}$$

4. For any real function $y(x) \in \dot{C}_n(0, \infty)$

$$\int_0^{\infty} \frac{y^2}{x^{2n}}\, dx \leqslant \frac{2^{2n}}{[(2n-1)!!]^2} \int_0^{\infty} |y^{(n)}|^2\, dx. \tag{8}$$

5. For any real function $y(x) \in C_1[0, \infty]$ for any $\alpha > -1$

$$\int_0^x x^\alpha y^2\, dx \leqslant \frac{4}{(\alpha+1)^2} \int_0^{\infty} x^{\alpha+2} y'^2\, dx. \tag{9}$$

To prove inequality (3) we start from the obvious relation

$$\|y^{(m)}\|^2 = \int_{-\infty}^{\infty} |y^{(m)}|^2\,dx = \left|\int_{-\infty}^{\infty} y^{(m-1)}\,y^{(m+1)}\,dx\right| \leqslant \left\{\int_{-\infty}^{\infty} |y^{(m+1)}|^2\,dx\right\}^{\frac{1}{2}} \cdot \left\{\int_{-\infty}^{\infty} |y^{(m-1)}|^2\,dx\right\}^{\frac{1}{2}}.$$

Repeating the same procedure for estimating the integrals on the right-hand side of this inequality and so on, we obtain inequality (3).

To prove inequality (6) we shall start from the relation

$$\int_{-\infty}^{\infty} y'^2\,dx \leqslant \frac{1}{2}\int_{-\infty}^{\infty} (y''^2 + y^2)\,dx,$$

which is equivalent to the inequality

$$\int_{-\infty}^{\infty} (y'' + y)^2\,dx \geqslant 0.$$

This relation is identical to inequality (6) for $n = 2$. To prove relation (6) for any n it is sufficient to use in addition to (4) the inequalities which follow from it when y is replaced by $y', y'', \ldots, y^{(n-1)}$.

Inequality (7) follows from (3) and from the relation

$$\frac{M^2}{2} = \frac{1}{2}\left|\int_{\alpha}^{\infty} (y^2)'\,dx\right| = \left|\int_{\alpha}^{\infty} yy'\,dx\right| \leqslant$$

$$\leqslant \left\{\int_{\alpha}^{\infty} y^2\,dx\right\}^{\frac{1}{2}} \cdot \left\{\int_{\alpha}^{\infty} y'^2\,dx\right\}^{\frac{1}{2}} \leqslant \left\{\int_{-\infty}^{\infty} y'^2\,dx\right\}^{\frac{1}{2}} = \|y'\|,$$

where the number α is determined by the condition $|y(\alpha)| = M$.

To prove inequality (8) we represent the integral on the left-hand side of this inequality in the form

$$\int_0^{\infty} x^{-2n}y^2\,dx = 2\int_0^{\infty} x^{-2n}\,dx \int_0^{x} y(t)\,y'(t)\,dt =$$

$$= 2\int_0^{\infty} y(t)\,y'(t)\,dt \int_t^{\infty} x^{-2n}\,dx = \frac{2}{2n-1}\int_0^{\infty} t^{1-2n}y(t)\,y'(t)\,dt.$$

Hence by means of the Cauchy-Bunyakovskii inequality we obtain

$$\int_0^{\infty} x^{-2n}y^2\,dx \leqslant \frac{2}{2n-1}\left\{\int_0^{\infty} x^{-2n}y^2\,dx\right\}^{\frac{1}{2}}\left\{\int_0^{\infty} x^{2-2n}y'^2\,dx\right\}^{\frac{1}{2}},$$

leading to the inequality

$$\int_0^{\infty} \frac{y^2}{x^{2n}}\,dx \leqslant \frac{2^2}{(2n-1)^2}\int_0^{\infty} \frac{y'^2}{x^{2n-2}}\,dx.$$

Repeating the above procedure once more for the integral on the right-hand side of the last inequality and so on, we obtain inequality (8).

Finally, to prove inequality (9), we use the representation

$$\int_0^\infty x^\alpha y^2\,dx = -2\int_0^\infty y(t)\,y'(t)\,dt\int_0^t x^\alpha\,dx,$$

from which by means of the Cauchy-Bunyakovskii inequality we obtain the relation

$$\int_0^\infty x^\alpha y^2\,dx \leqslant \frac{2}{\alpha+1}\left\{\int_0^\infty x^\alpha y^2\,dx\right\}^{\frac{1}{2}}\left\{\int_0^\infty x^{\alpha+2}y'^2\,dx\right\}^{\frac{1}{2}},$$

which is equivalent to inequality (9).

In connection with the inequalities of the present subsection see also /100/. We note that inequality (5) is a particular case of the general inequalities of Yu.I. Lyubich for the powers of a linear operator in a linear normalized space /64 (2)/.

27. Criteria for the boundedness and discreteness of the negative part of the spectrum. The simplest criterion of this kind is given by

Theorem 1 /31 (3)/. If

$$\lim_{x\to\infty} q^*(x) = 0,$$

then on the negative λ-semi-axis the spectrum of the operator $\tilde{L}$ contains the set of eigenvalues bounded from below with $\lambda = 0$ as the only possible limit point.

Proof. Let us choose the decomposition point $\gamma = \gamma_\varepsilon (\varepsilon > 0)$, so that for $x > \gamma$ we have $q(x) > -\varepsilon$. If the corresponding decomposition of the operator $\tilde{L}$ is determined by the relations

$$\tilde{L} = \tilde{L}_0, \quad L_0 = L_1 \oplus L_\gamma, \tag{10}$$

then, according to the decomposition principle, $C(\tilde{L}) = C(L_\gamma)$.

Furthermore, since for any $y \in \mathfrak{D}_{L_\gamma}$ we have the inequality

$$(L_\gamma y,\ y) = \int_\gamma^\infty |y^{(n)}|^2\,dx + \int_\gamma^\infty q(x)\,|y|^2\,dx > -\varepsilon\|y\|^2,$$

the operator L_γ is bounded from below by the number $-\varepsilon$, so that $C(\tilde{L}_\gamma)\cap(-\infty,\ -\varepsilon) = \emptyset$, and, consequently, we also have $C(\tilde{L})\cap(-\infty,\ -\varepsilon) = \emptyset$. Therefore, since the number $\varepsilon > 0$ is arbitrary it follows that

$$C(\tilde{L})\cap(-\infty,\ 0) = \emptyset.$$

Finally, since the operator L_γ is bounded from below, it follows that the operator $\tilde{L}$ is bounded from below. The theorem is thus proved.

Theorem 2 /31 (9)/. If the inequality

$$\int_{M_\delta} |q^*(x)|\,dx < \infty, \tag{11}$$

is satisfied for any $\delta > 0$ where M_δ is the set of values of x for which $|q^*(x)| \geqslant \delta$, then the negative part of the spectrum of the operator $\tilde{L}$ is discrete and bounded.

Proof. By virtue of Theorem 28 of No. 10 it is sufficient to show that for every $\varepsilon > 0$ there exists a number $N = N_\varepsilon$, such that the quadratic functional

$$\Phi[y] = \int_N^\infty |y^{(n)}|^2 dx + \int_N^\infty q^*(x) |y|^2 dx + \varepsilon \int_N^\infty |y|^2 dx \qquad (12)$$

is non-negative for any N-finite function $y \in D_L$.

For a given positive ε we split up the semi-axis $J = [0, \infty)$, into two disjoint sets

$$J = \mathfrak{M} + \mathfrak{N},$$

defined by the conditions

$$|q(x)| < \tfrac{\varepsilon}{2} \quad \text{for} \quad x \in \mathfrak{M},$$

$$|q(x)| \geqslant \tfrac{\varepsilon}{2} \quad \text{for} \quad x \in \mathfrak{N}.$$

We now introduce the following relation

$$J_N = [N, \infty), \quad \mathfrak{M}_N = J_N \cap \mathfrak{M}, \quad \mathfrak{N}_N = J_N \cap \mathfrak{N}$$

and choose the number $N = N_\varepsilon$ so that

$$\int_{\mathfrak{N}_N} |q^*(x)| \, dx < \min \left\{ \frac{1}{2}, \frac{\varepsilon^{4n-2}}{4^n} \right\}. \qquad (13)$$

Since the functional (12) is homogeneous, it may be assumed that

$$\int_N^\infty |y|^2 dx = 1. \qquad (14)$$

We obviously have

$$\int_{\mathfrak{M}_N} q^*(x) |y|^2 dx + \tfrac{\varepsilon}{2} \int_{\mathfrak{M}_N} |y|^2 dx \geqslant 0,$$

so that

$$\Phi_\varepsilon[y] \geqslant \int_N^\infty |y^{(n)}|^2 dx + \int_{\mathfrak{N}_N} q^*(x) |y|^2 dx + \tfrac{\varepsilon}{2} \int_N^\infty |y|^2 dx.$$

Let M denote the maximum value of the finite function $|y(x)|$ on the closed set $\mathfrak{N}_N$. If $M < \varepsilon$, then using normalization (14) and condition (13) we obtain

$$\Phi_\varepsilon[y] \geqslant \int_{\mathfrak{N}_N} q^*(x) |y|^2 dx + \\ + \tfrac{\varepsilon}{2} \int_N^\infty |y|^2 dx \geqslant \varepsilon \left\{ \frac{1}{2} - \int_{\mathfrak{N}_N} |q^*(x)| \, dx \right\}.$$

If on the other hand, $M \geqslant \varepsilon$, then from inequality (7) and condition (13) we have

$$\Phi_\varepsilon[y] \geqslant \int_N^\infty |y^{(n)}|^2 dx - \int_{\mathfrak{N}_N} |q^*(x)| \, |y|^2 dx \geqslant \\ \geqslant M^2 \left\{ \frac{1}{4^n} M^{4n-2} - \int_{\mathfrak{N}_N} |q^*(x)| \, dx \right\} \geqslant 0.$$

Thus, in both cases $\Phi_\delta[y] > 0$, Q. E. D.

From Theorem 2 we obtain Theorem 1, and, in particular, two simple criteria for the discreteness and boundedness of the negative part of the spectrum of the operator $\tilde{L}$. These criteria may be expressed by the relation

$$\lim_{x\to\infty} q(x) = 0$$

or

$$\int_0^\infty |q(x)|\,dx < \infty.$$

The first of these criteria was essentially established for the case when $n=1$ by H. Weyl. This was done by means of oscillation theorems (see /20 (2)/). The second criterion too was originally established for $n=1$ by H. Weyl /20 (2)/ who made use of asymptotic formulas (see also /61 (1)/).

Corollary. If for some value $r \geqslant 1$

$$\int_0^\infty |q^*(x)|^r\,dx < \infty,$$

then the negative part of the spectrum of the operator $\tilde{L}$ is discrete and bounded from below.

In fact, for $x \in M_\delta$ we have

$$\frac{1}{\delta}|q^*(x)| \geqslant 1$$

and consequently

$$\frac{1}{\delta}|q^*(x)| \leqslant \frac{1}{\delta^r}|q^*(x)|^r,$$

so that

$$\int_{M_\delta} |q^*(x)|\,dx \leqslant \frac{1}{\delta^{r-1}} \int_{M_\delta} |q^*(x)|^r\,dx < \infty,$$

thus condition (11) is fulfilled.

This corollary with $q^*(x)$ replaced by $q(x)$ was originally established for $r=1$ by I. M. Rapoport in /82 (1)/ who used asymptotic methods.

The following result, due to M. Sh. Birman, shows that the requirement in Theorem 1 that the negative part of the potential tend to zero may be replaced by the condition that its average value converge to zero.

Theorem 3 /13 (5)/. If for some (and, consequently, for any) $\omega \neq 0$ *we have*

$$\lim_{x\to\infty} \int_x^{x+\omega} q^*(t)\,dt = 0, \tag{15}$$

then the negative part of the spectrum of the operator $\tilde{L}$ *is bounded from below and discrete.*

Proof. It is sufficient to show that the functional (12) is non-negative. That is, we must show the inequality

$$\int_N^\infty |q^*(x)| \cdot |y|^2 dx < \int_N^\infty |y^{(n)}|^2 dx + \varepsilon \int_N^\infty |y|^2 dx \tag{16}$$

for any $\varepsilon > 0$ for all N-finite functions $y \in \mathfrak{D}_L$.

Let $\eta > 0$, and let the number $N = N_\eta$ be so chosen that for $x > N$ the following inequality is satisfied

$$\int_x^{x+\omega} |q^*(t)|\, dt < \eta.$$

Let $y(x)$ be any real N-finite function of $\mathfrak{D}_L$.

We split up the semi-axis $x > N$ into segments Ω_k of length ω. On each of these segments we determine the point x_k by the following condition

$$y^2(x_k) = \frac{1}{\omega} \int_{\Omega_k} y^2(t)\, dt.$$

To do this, we make use of the theorem on the average value.

With this choice of the points x_k and by virtue of the estimate

$$|y^2(x) - y^2(x_k)| = 2\left|\int_{x_k}^{x} y'y\, dt\right| \leqslant \int_{\Omega_k} (y'^2 + y^2)\, dt,$$

valid on Ω_k, we obtain

$$\int_{\Omega_k} |q^*(x)|\, y^2\, dx = \int_{\Omega_k} |q^*(x)|\, [y^2(x) - y^2(x_k)]\, dx + \\ + \int_{\Omega_k} |q^*(x)|\, y^2(x_k)\, dx < \eta \int_{\Omega_k} (y'^2 + y^2)\, dt + \frac{\eta}{\omega} \int_{\Omega_k} y^2\, dt$$

or, considering $\omega > 1$

$$\int_{\Omega_k} |q^*(x)|\, y^2\, dx < 2\eta \int_{\Omega_k} (y'^2 + y^2)\, dx.$$

Summing over k on both sides of this inequality, we obtain

$$\int_N^\infty |q^*(x)|\, y^2\, dx < 2\eta \int_N^\infty (y'^2 + y^2)\, dx. \tag{17}$$

From (17) and inequality (6) of the previous subsection follows the relation

$$\int_N^\infty |q^*(x)|\, y^2\, dx < 2\eta(1 + C_n) \int_N^\infty (y^{(n)^2} + y^2)\, dx. \tag{18}$$

From this relation, when $2\eta(1 + C_n) < \varepsilon < 1$ we obtain the required inequality (16), and the theorem is proved.

Inequality (18) established above implies more than actually required in the statement of the theorem, namely the boundedness from below and the discreteness of the negative part of the spectrum of the operator $\tilde{L}_h$, generated by operation (2) for any $h > 0$. That is, the spectrum of the operator $\tilde{L}$ is h-stable discrete and bounded from below.

As M. Sh. Birman points out, for a negative potential, condition (15) is not only sufficient but also necessary in order that the negative part of the spectrum of the operator $\tilde{L}$ be h-stable discrete and bounded.

Thus we have

Theorem 4/14(5)/. In order that the negative part of the spectrum of the operator $\tilde{L}$ be h-stable bounded from below and discrete, it is sufficient, and in the case $q(x) \leqslant 0$ also necessary, that condition (15) is fulfilled.

Proof of the sufficiency was given above. To prove the necessity we assume that condition (15) is not fulfilled, that is, for some infinite sequence of points $x_k \to \infty$, and for some $\omega > 0$,

$$\int_{x_k}^{x_k+\omega} |q(x)|\,dx > \rho > 0.$$

Let us construct a function $\varphi \in \mathfrak{D}_L$ which satisfies the conditions

$$\varphi(x) = \begin{cases} 1 & (0 \leqslant x \leqslant \omega). \\ 0 & (x \leqslant -\delta,\ x \geqslant \omega + \delta). \end{cases}$$

By means of the function φ, we construct the system of finite functions

$$\varphi_k(x) = \varphi(x - x_k) \tag{19}$$

having the disjoint supports $[x_k - \delta,\ x_k + \omega + \delta]$.

For this system of functions the integral

$$\int_0^\infty (\varphi_k^{(n)^2} + \varphi_k^2)\,dx = C$$

does not depend on k, while

$$\int_0^\infty |q(x)| \cdot |\varphi_k|^2 dx > \int_{x_k}^{x_k+\omega} |q(x)|\,dx > \rho > 0.$$

Thus, for any N there exists an N-finite function $\varphi_k(x)(k = k_N)$, for which

$$\int_0^\infty |q(x)|\,|y|^2 dx > \frac{\rho}{C} \int_0^\infty (y^{(n)^2} + y^2)\,dx, \tag{20}$$

hence, by virtue of Theorem 28 of No. 10, for $h \leqslant \frac{\rho}{C}$ it follows that $C(\tilde{L}_h) \cap (-\infty, 0) \neq 0$. The theorem is therefore proved.

28. Criteria for the semi-boundedness and discreteness of the whole spectrum. The simplest criterion of this kind is given in the following.

Theorem 5 /31 (1)/. If

$$\lim_{x \to +\infty} q(x) = +\infty, \tag{21}$$

then the spectrum of the operator $\tilde{L}$ is bounded from below and discrete.

Proof. From Theorem 1 of the previous subsection it follows that the spectrum $S(\tilde{L})$ is bounded from below. To prove that $S(\tilde{L})$ is discrete, we choose the decomposition point $\gamma = \gamma_N$ so that for $x > \gamma$ we have $q(x) > N$. If the corresponding decomposition of the operator $\tilde{L}$ is determined by relations (10), then, according to the decomposition principle, we have $C(\tilde{L}) = C(L_\gamma)$.

Furthermore, since for any $y \in \mathfrak{D}_{L_\gamma}$ we have the inequality

$$(L_\gamma y, y) = \int_\gamma^\infty |y^{(n)}|^2 dx + \int_\gamma^\infty q(x)|y|^2 dx > N\|y\|^2,$$

then the operator L_γ is bounded from below by the number N, so that $C(L_\gamma) \cap (-\infty, N) = \emptyset$ and consequently $C(\tilde{L}) \cap (-\infty, N) = \emptyset$. Hence, since the number N is arbitrary, it follows that $C(\tilde{L}) = \emptyset$ and the theorem is proved.

For $n = 1$ Theorem 5 coincides with the criterion established for the first time by H. Weyl /20·(2)/, using Sturm's comparison theorems and assuming monotonic potential. This theorem was generalized later to non-monotonic potentials by E. Titchmarsch /95(2)/.

As was shown by A.M. Molchanov /70/, condition (21) requiring the potential to tend to infinity can be replaced by a condition requiring its average value to tend to infinity, providing we assume the potential to be bounded from below. The condition of A.M. Molchanov is not only sufficient but also necessary for the discreteness of the spectrum in the class of potentials bounded from below. Thus, as it follows from Theorem 2, the condition of the boundedness from below of the potential $q(x)$ is not necessary for the semi-boundedness from below of the spectrum of the operator $\tilde{L}$.

A.M. Molchanov's theorem is established below by means of the decomposition method for the two-term operation (1) of any order.

Theorem 6 /70/. If the potential $q(x)$ is bounded from below, then a necessary and sufficient condition for the spectrum of the operator $\tilde{L}$ to be discrete is that the limit relation

$$\lim_{x\to\infty} \int_x^{x+\omega} q(t)\,dt = +\infty \tag{22}$$

is satisfied for any $\omega > 0$.

Proof. Without loss of generality, we may assume

$$q(x) \geqslant 1 \qquad (x \geqslant 0). \tag{23}$$

Suppose relation (22) holds. To prove that the spectrum of the operator $\tilde{L}$ is discrete it is sufficient to show that the negative part of the spectrum of the operator $\tilde{L} - \mu I$ is discrete for any $\mu > 0$. According to Theorem 28 of No. 10 this is equivalent to verifying the inequality

$$\mu \int_N^\infty y^2 dx < \int_N^\infty (|y^{(n)}|^2 + q|y|^2)\,dx \tag{24}$$

on all the N-finite functions $y \in \mathfrak{D}_L$ for some N.

We choose the number $N = N_\omega$ for a given ω so that for $x > N$ we have the inequality

$$\int_x^{x+\omega} q(t)\,dt > 1. \tag{25}$$

and let $y(x)$ be any N-finite function of $\mathfrak{D}_L$.

As in the proof of Theorem 3, we split up the semi-axis $x > N$ into segments Ω_k of length ω, and on each such segment determine the point x_k

by the condition

$$y^2(x_k) = \frac{\int\limits_{\Omega_k} q(x)\, y^2(x)\, dx}{\int\limits_{\Omega_k} q(x)\, dx}. \tag{26}$$

By virtue of inequality (23) we have for $x \in \Omega_k$ that

$$|y^2(x) - y^2(x_k)| = 2\left|\int\limits_{x_k}^{x} yy'\, dt\right| \leqslant$$
$$\leqslant \int\limits_{\Omega_k} (y'^2 + y^2)\, dt \leqslant \int\limits_{\Omega_k} (y'^2 + qy^2)\, dt,$$

hence, by choosing the points x_k according to relation (26) we obtain the inequality

$$\int\limits_{\Omega_k} y^2\, dx = \int\limits_{\Omega_k} y^2(x_k)\, dx + \int\limits_{\Omega_k} [y^2(x) - y^2(x_k)]\, dx \leqslant$$
$$\leqslant \omega \frac{\int\limits_{\Omega_k} q(x)\, y^2(x)\, dx}{\int\limits_{\Omega_k} q(x)\, dx} + \omega \int\limits_{\Omega_k} (y'^2 + qy^2)\, dx,$$

or, by virtue of (25)

$$\int\limits_{\Omega_k} y^2\, dx \leqslant \omega \int\limits_{\Omega_k} qy^2\, dx + \omega \int\limits_{\Omega_k} (y'^2 + qy^2)\, dx.$$

Summing both sides of the last inequality over k, we obtain

$$\int\limits_{N}^{\infty} y^2\, dx \leqslant \omega \int\limits_{N}^{\infty} qy^2\, dx +$$
$$+ \omega \int\limits_{N}^{\infty} (y'^2 + qy^2)\, dx < 2\omega \int\limits_{N}^{\infty} (y'^2 + qy^2)\, dx. \tag{27}$$

From (27), (23) and (6) we finally have

$$\int\limits_{N}^{\infty} y^2\, dx < 2\omega(C_n + 1) \int\limits_{N}^{\infty} (y^{(n)2} + qy^2)\, dx.$$

Inequality (24) follows on taking $\omega < \frac{1}{2\mu(C_n + 1)}$.

Passing now to the proof of the necessity of condition (22), we assume that the spectrum $S(\tilde{L})$ is discrete, but for some sequence of points $x_k \to \infty$ for some $\omega > 0$

$$\int\limits_{x_k}^{x_k + \omega} q(t)\, dt < \rho \qquad (\rho < \infty).$$

As in the proof of Theorem 4, let us construct a real function $\varphi \in \mathfrak{D}_L$, satisfying the conditions

$$\varphi(x) = \begin{cases} 1 & (\delta \leqslant x \leqslant \omega - \delta), \\ 0 & (x \leqslant 0,\ x \geqslant \omega). \end{cases} \qquad |\varphi(x)| \leqslant 1,$$

and by means of it, we construct the system of finite functions

$$\varphi_k(x) = \varphi(x - x_k)$$

having the disjoint supports $[x_k - \delta,\ x_k + \omega + \delta]$. For this system of functions we have

$$\int_0^\infty (\varphi_k^{(n)2} + q\varphi_k^2)\,dx < C + \int_{x_k}^{x_k+\omega} q(x)\,dx < C + \rho,$$

whereas

$$\int_0^\infty \varphi_k^2\,dx > \omega - 2\delta.$$

Thus, for any N there exists an N-finite function $\varphi_k \in \mathfrak{D}_L$ $(k = k_N)$, for which

$$\int_N^\infty y^2\,dx > \frac{\omega - 2\delta}{C + \rho} \int_N^\infty (y^{(n)2} + qy^2)\,dx,$$

therefore, by virtue of Theorem 28 of No. 10, it follows for $\mu > \frac{C+\rho}{\omega - 2\delta}$ that $C(\tilde{L} - \mu I) \cap (-\infty,\ 0) \neq 0$.

The theorem is now completely proved.

Under the conditions of Theorem 5, the eigenvalues of the operator L increase with increase in the potential $q(x)$. With $n = 1$, $q(x) \to \infty$, $q'(x) > 0$, and $q''(x) > 0$ de Wet and Mandl /21/ obtained an asymptotic formula for the number $N(\lambda)$ of eigenvalues smaller than λ. V. A. Tkachenko drew the author's attention to the fact that their proof can be applied without modifications to operation (1) of any order, and leads, for $\lambda \to \infty$, to the asymptotic formula

$$N(\lambda) \sim \frac{1}{\pi} \int_{q(x) < \lambda} \sqrt[2n]{\lambda - q(x)}\,dx.$$

To derive this formula we divide the interval $(0,\ a)$ into sub-intervals by the points

$$0 = x_0 < x_1 < \ldots < x_{m-1} < x_m = a$$

where $q(a) = \lambda$. In each of these sub-intervals the potential $q(x)$ is replaced once by $q(x_k)$, and then by $q(x_{k-1})$. On the ends of the partial intervals the following conditions are given: $y^{(r)}(x_k) = 0$ and $y^{(n+r)}(x_k) = 0$, respectively $(k = 0,\ 1,\ \ldots\ m;\ r = 0,\ 1,\ \ldots,\ n-1)$. The application of Courant's variational principle yields the upper and lower limits

$$\frac{1}{\pi} \sum_{k=1}^{m} (x_k - x_{k-1}) \sqrt[2n]{\lambda - q(x_k)} - m \leqslant N(\lambda) \leqslant$$

$$\leqslant \frac{1}{\pi} \sum_{k=1}^{m} (x_k - x_{k-1}) \sqrt[2n]{\lambda - q(x_{k-1})} + m.$$

Furthermore by choosing the division into sub-intervals so that in each of them the oscillation of the function $\sqrt[2n]{\lambda - q(x)}$ equals $\frac{1}{m}\sqrt[2n]{\lambda - q(0)}$, we then have the estimate

$$\left| N(\lambda) - \frac{1}{\pi} \int_{q(x) < \lambda} \sqrt[2n]{\lambda - q(x)}\,dx \right| \leqslant \frac{a\sqrt[2n]{\lambda - q(0)}}{\pi m} + m.$$

Minimizing the right-hand side of this estimate with respect to m we find the inequality

$$\left| N(\lambda) - \frac{1}{\pi} \int_{q(x) < \lambda} \sqrt[2n]{\lambda - q(x)}\,dx \right| \leqslant 2\sqrt{\frac{a}{\pi}\sqrt[2n]{\lambda - q(0)}},$$

thus by means of the condition $q''(x) > 0$ we obtain the above-mentioned asymptotic formula for $N(\lambda)$.

We note that the requirement $q''(x) > 0$ excludes some "boundary" zone of potentials from the domain of application of the asymptotic law. These potentials, however, satisfy the discreteness criterion (21), but increase more slowly than The existence of zones in which the asymptotic distribution of the eigenvalues is unknown is characteristic of singular problems.

29. Spectrum discreteness criteria for the case of the polar differential operation. The problem of finding the eigenfrequencies of the free-oscillations of a semi-infinite string of density $\rho = \rho(x)$ reduces to the differential equation

$$-y'' - \lambda\rho(x)\, y = 0 \qquad (x \geqslant 0).$$

Let us consider the more general equation

$$(-1)^n y^{(2n)} - \lambda\rho(x)\, y = 0 \qquad (x \geqslant 0), \tag{28}$$

where $\rho(x)$ is an arbitrary measurable positive function. The problem of finding the eigenvalues of the operator $\tilde{L}$, generated in the space $\mathcal{L}^2_\rho(0, \infty)$ by the operation

$$(-1)^n \frac{1}{\rho(x)} \frac{d^{2n}}{dx^{2n}}, \tag{29}$$

henceforth called polar, leads to equation (28). It is obvious that the construction of the operator $\tilde{L}$, together with the other procedures and propositions of § 2 of Chapter I, can easily be carried over to the case of the polar operation (29). In particular (see note to Theorem 28 of No. 10), a necessary and sufficient condition for the discreteness of the spectrum $S(\tilde{L})$ is that for any $\mu > 0$ there exists a number $N = N_\mu$, such that for all the N-finite functions $y \in \mathfrak{D}_L$, the following inequality holds

$$\int_N^\infty |y^{(n)}|^2\, dx - \mu \int_N^\infty \rho(x) |y|^2\, dx > 0. \tag{30}$$

Condition (30) obtained by the method of decomposition also enables us to find the criterion for the discreteness of the spectrum which was established in a different manner for $n = 1$ by I.S. Kats and M.G. Krein /50/. From this criterion it follows in particular that the frequency spectrum of a semi-infinite string may be discrete only when the mass of the string is finite. The following theorem concerning the discreteness criterion is due to M.Sh. Birman.

Theorem 7. A necessary and sufficient condition for the spectrum of the operator $\tilde{L}$, generated by the polar operation (29), to be discrete, is that the limit relation

$$\lim_{x \to \infty} x^{2n-1} \int_x^\infty \rho(t)\, dt = 0. \tag{31}$$

is satisfied.

Proof. Suppose relation (31) is satisfied. Then for a given $\varepsilon > 0$ there exists a number $N = N_\varepsilon$, such that for $x > N$ we have

$$x^{2n-1} \int_x^\infty \rho(t)\, dt < \varepsilon.$$

For any N-finite function $y \in \mathfrak{D}_L$ we therefore have

$$\int_N^\infty \rho y^2 dx = 2\int_N^\infty \rho\, dx \int_N^x yy'\, dt = 2\int_N^\infty yy'\, dt \int_t^\infty \rho\, dx =$$

$$= 2\int_N^\infty \frac{yy'}{t^{2n-1}} \cdot t^{2n-1} dt \int_t^\infty \rho\, dx \leqslant 2\varepsilon \int_N^\infty \frac{|yy'|}{t^{2n-1}} dt, \qquad (32)$$

but

$$\int_N^\infty \frac{|yy'|}{t^{2n-1}} dt \leqslant \left\{\int_N^\infty \frac{y^2}{t^{2n}} dt\right\}^{\frac{1}{2}} \left\{\int_N^\infty \frac{y'^2}{t^{2n-2}} dt\right\}^{\frac{1}{2}},$$

and, estimating each factor on the right-hand side of the last relation by means of inequality 4 of No. 26, we obtain

$$\int_N^\infty \frac{|yy'|}{t^{2n-1}} dt \leqslant \frac{2^n}{(2n-1)!!} \cdot \frac{2^{n-1}}{(2n-3)!!} \int_N^\infty |y^{(n)}|^2 dt. \qquad (33)$$

From (32) and (33) follows the inequality

$$\int_N^\infty \rho y^2 dx \leqslant \varepsilon C \int_N^\infty |y^{(n)}|^2 dx,$$

where the constant C depends only on n. Choosing $\varepsilon < \frac{1}{C\mu}$, we obtain inequality (30) from (33), from which, since the number $\mu > 0$ is arbitrary, it follows that the spectrum $S(\tilde{L})$ is discrete,

Turning to the proof of the necessity of condition (31), we assume that for some sequence of points $x_k \to \infty$, we have

$$x_k^{2n-1} \int_{x_k}^\infty \rho(t)\, dt > \sigma > 0.$$

For sufficiently large $\xi_k > x_k$ we then also have

$$x_k^{2n-1} \int_{x_k}^{\xi_k} \rho(t)\, dt > \sigma > 0,$$

and, passing if necessary to the subsequence, we may take $x_{k+1} > 2\xi_k$ for all k.

To construct the proper test system of finite functions with disjoint supports, we first choose some $2n$-fold continuously differentiable function $\psi(x)$ which is defined in the interval $[0, 1]$, and satisfies on its ends the conditions

$$\psi(0) = \psi'(0) = \ldots = \psi^{(2n)}(0) = 0.$$

$$\psi(1) = 1, \quad \psi'(1) = \psi''(1) = \ldots = \psi^{(2n-1)}(1) = 0.$$

Next, we set

$$\chi_k(x)=\begin{cases} 0 & x<x_k-\frac{1}{3}(x_k-\xi_{k-1}) \\ \psi\left(1+\frac{3(x-x_k)}{x_k-\xi_{k-1}}\right) & x_k-\frac{1}{3}(x_k-\xi_{k-1})\leq x<x_k, \\ 1 & x_k<x<\xi_k, \\ \psi\left(1-\frac{3(x-\xi_k)}{x_{k+1}-\xi_k}\right) & \xi_k<x<\xi_k+\frac{1}{3}(x_{k+1}-\xi_k), \\ 0 & x>\xi_k+\frac{1}{3}(x_{k+1}-\xi_k) \end{cases}$$

and finally, we form the system of finite functions with disjoint supports

$$\varphi_k(x)=\chi_k(x)\,x_k^{\frac{2n-1}{2}} \qquad (k=1,\ 2,\ \ldots).$$

This system belongs to $\mathfrak{D}_L$. For this system we have

$$\int_0^\infty \rho\varphi_k^2(x)\,dx \geqslant \int_{x_k}^{\xi_k} \rho\varphi_k^2(x)\,dx = x_k^{2n-1}\int_{x_k}^{\xi_k}\rho\,dx > \sigma > 0. \tag{34}$$

On the other hand,

$$\int_0^\infty |\varphi_k^{(n)}(x)|^2\,dx = x_k^{2n-1}\int_0^\infty |\chi_k^{(n)}(x)|^2\,dx =$$

$$= 3^{2n-1}x_k^{2n-1}\left\{\int_0^1 (x_k-\xi_{k-1})^{1-2n}\,|\psi^{(n)}(t)|^2\,dt + \right.$$

$$\left. + \int_0^1 (x_{k+1}-\xi_k)^{1-2n}\,|\psi^{(n)}(t)|^2\,dt\right\},$$

and since

$$\frac{x_k}{x_k-\xi_{k-1}}=\frac{1}{1-\frac{\xi_{k-1}}{x_k}}<2,$$

$$\frac{x_k}{x_{k+1}-\xi_k}<\frac{x_k}{\xi_k}<1,$$

we have for all k

$$\int_0^\infty |\varphi_k^{(n)}(x)|^2\,dx < C_n, \tag{35}$$

where the constant C_n is independent of k.

From (34) and (35) we obtain the following inequality for any k.

$$\int_0^\infty \rho\varphi_k^2(x)\,dx > \frac{\sigma}{C_n}\int_0^\infty |\varphi_k^{(n)}(x)|^2\,dx.$$

Thus for any N there exists a finite function $\varphi_k(x)\in\mathfrak{D}_L$ $(k=k_N)$, for which inequality (30) is not satisfied when $\mu=\sigma C_n^{-1}$. This contradicts the discreteness of the spectrum of the operator $\tilde{L}$, and the theorem is therefore proved.

Some other results concerning the spectrum of a singular string are contained in /49/.

30. Criteria for non-oscillatory and oscillatory behavior. By virtue of the results of No. 12, the criteria for the non-oscillatory or oscillatory

behavior of the differential equation

$$(-1)^n y^{(2n)} + q(x)\, y = 0 \tag{36}$$

are at the same time the criteria for the set of points of the negative part of the spectrum of the operator $\tilde{L}$ to be finite or infinite, respectively.

The theorems given below are formulated as criteria for non-oscillatory or oscillatory behavior. The following integral criterion for non-oscillatory behavior follows directly from inequality (8) of No. 26.

Theorem 8 /31 (11)/. If

$$\limsup_{x\to\infty} x^{2n-1} \int_x^\infty |q^*(t)|\, dt < \frac{\alpha_n^2}{2n-1}, \tag{37}$$

where

$$\alpha_n = \frac{(2n-1)!!}{2^n}, \tag{38}$$

then equation (36) is non-oscillatory.

Proof. By virtue of (37), we have for any N-finite function $y \in \mathfrak{D}_L$ that

$$\int_N^\infty |q^*(x)|\, y^2\, dx = 2\int_N^\infty |q^*(x)|\, dx \int_N^x yy'\, dt =$$

$$= 2\int_N^\infty yy'\, dt \int_t^\infty |q^*(x)|\, dx \leqslant 2\int_N^\infty \frac{|yy'|}{t^{2n-1}}\, t^{2n-1} \int_t^\infty |q^*(x)|\, dx \leqslant$$

$$\leqslant \frac{2\alpha_n^2}{2n-1} \int_N^\infty \frac{|yy'|}{t^{2n-1}}\, dt.$$

thus, on the basis of inequality (33), we obtain

$$\int_N^\infty |q^*(x)|\, y^2\, dx \leqslant \int_N^\infty |y^{(n)}|^2\, dx,$$

leading to the inequality

$$\int_N^\infty (|y^{(n)}|^2 + qy^2)\, dx \geqslant \int_N^\infty |y^{(n)}|^2\, dx - \int_N^\infty |q^*(x)|\, y^2\, dx \geqslant 0.$$

Hence by virtue of Theorem 28 of No. 10, it follows that equation (36) is non-oscillatory.

The following theorem is a direct generalization of Knezer's classical theorem on the oscillatory character of the solutions of the second-order differential equation.

Theorem 9 /31 (9)/. Equation (36) is non-oscillatory if for sufficiently large x the inequality

$$q(x) \geqslant -\alpha_n^2 x^{-2n}, \tag{39}$$

is satisfied. It is oscillatory if for sufficiently large x and some $\delta > 0$ the inequality

$$q(x) < -(\alpha_n^2 + \delta)\, x^{-2n}, \tag{40}$$

where Knezer's constant α_n^2, determined by (38), is satisfied.

Proof. From the results of No. 12, it is sufficient to show that the "standard" equation

$$\Lambda[y] \equiv (-1)^n y^{(2n)} + \alpha x^{-2n} y = 0 \tag{41}$$

is non-oscillatory if $\alpha \geqslant -\alpha_n^2$, and oscillatory if $\alpha < -\alpha_n^2$.

The Euler equation (41) reduces to an equation with constant coefficients, and its solution can be directly investigated.

Another proof makes use of theorem 8, from which the non-oscillatory character of equation (41) follows directly for $\alpha \geqslant -\alpha_n^2$. The proof of the oscillatory character of this equation for $\alpha < -\alpha_n^2$ requires, as in the similar theorems of Nos. 27 – 29, the construction of a special system of finite test functions, or it may be obtained as a direct consequence of Theorem 12 given below.

Finally, a third proof given below in detail, goes as follows. By the substitution

$$y = x^{n-\frac{1}{2}} u, \quad x = \exp t \tag{42}$$

we transform the operation

$$\mu[y] = (-1)^n \frac{d^{2n} y}{dx^{2n}}$$

to the form

$$x^{-\left(n+\frac{1}{2}\right)} \sum_{k=0}^{2n} \beta_k \frac{d^{2n-k} u}{dt^{2n-k}}, \tag{43}$$

where β_k are some constant coefficients, and $\beta_0 = (-1)^n$, $\beta_{2n} = \alpha_n$.

Hence, for the standard operation $\Lambda[y]$ in the case of any N-finite function $y \in \mathfrak{D}_L$ we obtain the equality

$$(\Lambda[y], y) = \int_{N'}^{\infty} \left\{ (\alpha + \alpha_n^2) |u|^2 + \sum_{r=0}^{n-1} (-1)^{n-r} \beta_{2r} \left| \frac{d^{n-r} u}{dt^{n-r}} \right|^2 \right\} dt,$$

the right-hand side of which can be represented in the form

$$\int_{N'}^{\infty} P\left[l \frac{d}{dt}\right] u \cdot \bar{u}\, dt,$$

where

$$P(\lambda) = (-1)^n \beta_0 \lambda^{2n} + (-1)^{n-1} \beta_2 \lambda^{2(n-2)} + \ldots - \beta_{2n-2} \lambda^2 + (\alpha_n + \alpha). \tag{44}$$

Thus,

$$(\Lambda[y], y) = \left(P\left[l \frac{d}{dt}\right] u, u\right). \tag{45}$$

Furthermore, since the continuous part of the spectrum of the operator $P\left[l \frac{d}{dt}\right]$ coincides with the set of values of the polynomial $P(\lambda)$ for $-\infty < \lambda < \infty$, and the discrete part of the spectrum is a finite set, then according to (45), a necessary and sufficient condition for the non-oscillatory character of the operation $\Lambda[y]$ is that the polynomial (44) be non-negative on the λ-axis. As will be verified below, all the coefficients of the polynomial $P(\lambda)$ except, possibly, the last one are non-negative, that is

$$(-1)^{n-r} \beta_{2r} \geqslant 0 \qquad (r = 0, 1, \ldots, n-1). \tag{46}$$

so that the condition for the polynomial (44) to be non-negative reduces to the inequality

$$\alpha_n^2+\alpha\geqslant 0. \tag{47}$$

Thus, if inequality (47) is satisfied, equation (41) is non-oscillatory, and if it is not satisfied then equation (41) is oscillatory, Q. E. D.

It remains to prove inequality (46). For this purpose we note that according to (43) the number β_k is the coefficient of $x^{-\left(n+\frac{1}{2}\right)}$ in the expression $\mu\left[x^{n-\frac{1}{2}}\ln^k x\right]$, so that

$$\beta_k=(-1)^n\frac{d^{2n}}{dx^{2n}}\left\{x^{n-\frac{1}{2}}\ln^k x\right\}\Big|_{x=1}$$

and consequently, the sign of β_k is the same as the sign of the coefficient of z^{2n} in the expansion in powers z of the function

$$f(z)=(-1)^n(1+z)^{n-\frac{1}{2}}\ln^k(1+z).$$

To expand the function $f(z)$ in a power series we represent it in the form

$$f(z)=\frac{d^k\zeta}{ds^k},$$

where

$$\zeta(s)=(-1)^n(1+z)^s.$$

Now

$$\zeta(s)=(-1)^n\sum_{j=0}^{\infty}\binom{s}{j}z^j,$$

so that

$$f(z)=(-1)^n\sum_{j=0}^{\infty}z^j\frac{d^k}{ds^k}\binom{s}{j},$$

hence the sign of β_k is the same as that of

$$(-1)^n\frac{d^k}{ds^k}\binom{s}{j}$$

for $j=2n$, $s=n-\frac{1}{2}$.

Setting $\sigma=s-n+\frac{1}{2}$, we obtain

$$(-1)^n\frac{d^k}{ds^k}\binom{s}{2n}\Big|_{s=n-\frac{1}{2}}=\frac{(-1)^n}{(2n)!}\frac{d^k}{d\sigma^k}Q(\sigma)\Big|_{\sigma=0}, \tag{48}$$

where

$$Q(\sigma)=\left[\sigma^2-\frac{(2n-1)^2}{4}\right]\left[\sigma^2-\frac{(2n-3)^2}{4}\right]\ldots\left[\sigma^2-\frac{1}{4}\right], \tag{49}$$

and, consequently, the sign of β_k is the same as that of the coefficient of σ^k in the polynomial $(-1)^nQ(\sigma)$, that is

$$(-1)^{n-r}\beta_{2r}\geqslant 0,\ \beta_{2r+1}=0 \qquad (r=0,\ 1,\ \ldots,\ n-1),$$

Q. E. D.

Apart from Theorem 8, the following theorem, too, gives an integral criterion for non-oscillatory behavior.

Theorem 10 /31 (9)/. If the inequality

$$\int_{M_\delta} x^{2n-1}|q^*(x)|\,dx < \infty, \tag{50}$$

is satisfied for some $\delta > 0$, *where* M_δ *is the set of values of* x *for which*

$$x^{2n}|q^*(x)| \geqslant \alpha_n^2 - \delta,$$

then equation (36) is non-oscillatory.

Proof. Transforming operation (1) by means of substitution (42), we obtain for any N-finite function $y \in \mathfrak{D}_L$

$$(l[y],\ y) = \int_{N'}^{\infty} \left|\frac{d^n u}{dt^n}\right|^2 dt + \sum_{r=1}^{n-1} (-1)^{n-r}\beta_{2r} \int_{N'}^{\infty} \left|\frac{d^{n-r}u}{dt^{n-r}}\right|^2 dt + \\ + \int_{N'}^{\infty} \tilde{q}(t)|u|^2 dt + \alpha_n^2 \int_{N'}^{\infty} |u|^2\,dt,$$

where

$$\tilde{q}(t) = x^{2n}q(x) \qquad (x = \exp t).$$

Now it is sufficient to establish the non-negativeness of the functional

$$F[u] = \int_{N'}^{\infty} \left|\frac{d^n u}{dt^n}\right|^2 dt + \int_{N'}^{\infty} \tilde{q}^*(t)|u|^2 dt + \alpha_n^2 \int_{N'}^{\infty} |u|^2 dt,$$

where

$$\tilde{q}^*(t) = x^{2n}q^*(x) \qquad (x = \exp t).$$

As in the proof of Theorem 2 of No. 27, we divide the semi-axis $J = [0, \infty)$ into two disjoint sets

$$J = \mathfrak{M} + \mathfrak{N},$$

defined by the conditions

$$|\tilde{q}^*(t)| < \alpha_n^2 - \delta \quad \text{for} \quad x \in \mathfrak{M},$$
$$|\tilde{q}^*(t)| \geqslant \alpha_n^2 - \delta \quad \text{for} \quad x \in \mathfrak{N}.$$

Setting

$$J_{N'} = (N', \infty), \quad \mathfrak{M}_{N'} = J_{N'} \cap \mathfrak{M}, \quad \mathfrak{N}_{N'} = J_{N'} \cap \mathfrak{N}$$

and, using condition (50), we choose the number N' so that

$$\int_{\mathfrak{N}_{N'}} |\tilde{q}^*(t)|\,dt < \min\left\{1, \frac{\delta^{4n-2}}{4^n}\right\}. \tag{51}$$

Since the functional $F[u]$ is homogeneous, it may be assumed that

$$\int_{N'}^{\infty} |u|^2 dt = 1. \tag{52}$$

We obviously have

$$\int_{\mathfrak{M}_{N'}} q^*(t)|u|^2 dt + (\alpha_n^2 - \delta) \int_{\mathfrak{N}_{N'}} |u|^2 dt \geqslant 0,$$

so that

$$F[u] \geqslant \int_{N'}^{\infty} \left| \frac{d^n u}{dt^n} \right| dt + \int_{\mathfrak{N}_{N'}} \tilde{q}^*(t) |u|^2 dt + \delta \int_{N'}^{\infty} |u|^2 dt.$$

Let M denote the maximum value of the finite function $|u(t)|$ on the closed set $\mathfrak{N}_{N'}$.

If $M < \delta$, then from the normalization (52) and condition (51), we obtain

$$F[u] \geqslant \int_{\mathfrak{N}_{N'}} \tilde{q}^*(t) |u|^2 dt + \delta \int_{N'}^{\infty} |u|^2 dt \geqslant$$

$$\geqslant \delta \left\{ 1 - \int_{\mathfrak{N}_{N'}} |\tilde{q}^*(t)| dt \right\} > 0.$$

If $M \geqslant \delta$, then from inequality (7) of No. 26, and condition (51) it follows that

$$F[u] \geqslant \int_{N'}^{\infty} \left| \frac{d^n u}{dt^n} \right|^2 dt - \int_{\mathfrak{N}_{N'}} |\tilde{q}^*(t)| |u|^2 dt \geqslant$$

$$\geqslant M^2 \left\{ \frac{1}{4^n} M^{4n-2} - \int_{\mathfrak{N}_{N'}} |\tilde{q}^*(t)| dt \right\} \geqslant 0.$$

Thus, in both cases $F[u] \geqslant 0$, Q. E. D.

The conditions of Theorem 10 are fulfilled, in particular, if for some $r \geqslant 1$

$$\int_0^{\infty} x^{2nr-1} |q^*(x)|^r dx < \infty.$$

Setting $n = 1$, $r = 1$ we are led to the well-known criterion for the non-oscillatory character of the solutions of the second-order differential equation (see /10/).

The following theorem gives an integral criterion for the oscillatory character of equations (36) for negative potentials.

Theorem 11 /31 (9)/. If $q(x) \leqslant 0$ for large x and

$$\lim_{x \to \infty} \sup x^{2n-1} \int_x^{\infty} |q(t)| dt > A_n^2. \tag{53}$$

where

$$A_n^{-1} = \frac{\sqrt{2n-1}}{(n-1)!} \sum_{k=1}^{n} \frac{(-1)^{k-1} \binom{n-1}{k-1}}{2n-k},$$

then equation (36) is oscillatory.

Proof. It is sufficient to construct for any $N > 0$ an N-finite function $y \in \mathfrak{D}_L$, for which

$$\Phi[y] = \int_N^{\infty} |y^{(n)}|^2 dx + \int_N^{\infty} q(x) |y|^2 dx < 0.$$

On the basis of Lemma 5 of No. 10, the requirements for the function y to belong to the manifold $\mathfrak{D}_L$ can be weakened by assuming that the finite

function $y(x)$ is $2n$-fold continuously differentiable on the axis, with the exception of a finite number of points where it is assumed continuous together with its derivatives only up to the order $n-1$. To construct such a function $y(x)$, we first introduce the polynomial $\omega(z)$ of degree $2n-1$, determined by the conditions

$$\omega(0)=0, \quad \omega(1)=1, \quad \omega^{(k)}(0)=\omega^{(k)}(1)=0 \quad (k=1, \ldots, n-1). \tag{54}$$

We now assume that

$$\int_0^1 |\omega^{(n)}(z)|^2 dz = A_n^2. \tag{55}$$

and defer the proof until later on. We choose by means of relation (53) a sequence $\rho_k \to \infty$, for which we have for some $\alpha > 0$ that

$$\lim_{\rho_k \to \infty} \rho_k^{2n-1} \int_{\rho_k}^{\infty} |q(t)|\, dt \geqslant A_n^2 + \alpha.$$

We construct the function $y(x)$ according to the formulas

$$y(x) = \begin{cases} 0 & (x \leqslant \mu\rho), \\ \omega\left(\dfrac{x-\mu\rho}{\rho[1-\mu]}\right) & (\mu\rho < x \leqslant \rho), \\ 1 & (\rho < x \leqslant R), \\ \omega\left(\dfrac{\nu R - x}{R[\nu-1]}\right) & (R < x \leqslant \nu R), \\ 0 & x \geqslant \nu R \end{cases} \tag{56}$$

We first choose the parameter $\mu > 0$ so small that

$$\frac{A_n^2}{(1-\mu)^{2n-1}} < A_n^2 + \frac{\alpha}{4}.$$

We then choose the parameter $\rho = \rho_k (k = k_N)$ so that $\mu\rho < N$ and

$$\int_{\rho}^{\infty} |q(t)|\, dt > \frac{A_n^2 + \frac{\alpha}{2}}{\rho^{2n-1}}.$$

This is possible by virtue of (53). Finally, we choose the parameters R and ν so large that

$$\int_{\rho}^{R} |q(t)|\, dt > \frac{A_n^2 + \frac{\alpha}{2}}{\rho^{2n-1}}$$

and

$$\frac{A_n^2 \rho^{2n-1}}{R^{2n-1}(\nu-1)^{2n-1}} < \frac{\alpha}{4}.$$

The value of the functional $\Phi[y]$ on the constructed function y is then given by

$$\Phi[y] \leqslant \frac{A_n^2}{\rho^{2n-1}(1-\mu)^{2n-1}} + \frac{A_n^2}{R^{2n-1}(\nu-1)^{2n-1}} - \int_{\rho}^{R} |q(t)|\, dt =$$

$$= \frac{1}{\rho^{2n-1}} \left\{ \frac{A_n^2}{(1-\mu)^{2n-1}} + \frac{A_n^2 \rho^{2n-1}}{R^{2n-1}(\nu-1)^{2n-1}} - \rho^{2n-1} \int_{\rho}^{R} |q(t)|\, dt \right\} < 0.$$

Q. E. D.

It remains to prove equality (55). It is obvious that

$$\omega'(z) = Cz^{n-1}(z-1)^{n-1},$$

so that

$$\omega(z) = C\int_0^z z^{n-1}(z-1)^{n-1}\,dz,$$

and since we should have $\omega(1)=1$, then

$$C^{-1} = \sum_{k=1}^{n} \frac{(-1)^{k-1}\binom{n-1}{k-1}}{2n-k}. \tag{57}$$

Furthermore we have

$$\int_0^1 |\omega^{(n)}(z)|^2\,dz = \frac{C^2[(n-1)!]^2}{2}\int_{-1}^{1} |\mathscr{L}_{n-1}(\xi)|^2\,d\xi,$$

where $\mathscr{L}_{n-1}(\xi)$ is the Legendre polynomial

$$\mathscr{L}_{n-1}(\xi) = \frac{1}{2^{n-1}(n-1)!}\frac{d^{n-1}}{d\xi^{n-1}}(\xi^2-1)^{n-1},$$

so that

$$\int_{-1}^{1} |\mathscr{L}_{n-1}(\xi)|^2\,d\xi = \frac{2}{2n-1},$$

which together with (57) proves equality (55).

It is not accidental that the polynomial $\omega(z)$ was chosen in (56) as a smoothing function in the proof of the present and the following theorem. The natural tendency to obtain the smallest value of the constants A_n and B_n in (53) and (58) leads unambiguously to smoothing by means of the polynomial $\omega(z)$, determined by the interpolation conditions (54). Criterion (53) should be compared with the previously established criterion (37) for non-oscillatory behavior in order to get an idea of its accuracy.

For $n=1$, condition (53) takes the form

$$\limsup_{x\to\infty} x\int_x^{\infty} |q(t)|\,dt > 1,$$

and in this case Theorem 11 gives a criterion which was obtained earlier in a different way by E. Hille /105/.

The second part of Theorem 9 is refined by the following:

Theorem 12 /31 (9)/. If the inequality

$$q(x) + \alpha_n^2 x^{-2n} \leqslant 0$$

and the limit relation

$$\limsup_{r\to\infty} (\ln r)\int_r^{\infty} x^{2n-1}\,|q(x)+\alpha_n^2 x^{-2n}|\,dx = \infty, \tag{58}$$

are satisfied for large x, then equation (36) is oscillatory.

Proof. Transforming operation (1) by means of the substitution (42), we obtain for any N-finite function $y(x)\in\mathfrak{D}_L$ the equality

$$(l[y], y)=\int_{N'}^{\infty}\left|\frac{d^n u}{dt^n}\right|^2 dt+$$

$$+\sum_{r=1}^{n-1}(-1)^{n-r}\beta_{2r}\int_{N'}^{\infty}\left|\frac{d^{n-r}u}{dt^{n-r}}\right|^2 dt+$$

$$+\int_{N'}^{\infty}\tilde{q}(t)|u|^2 dt+\alpha_n^2\int_{N'}^{\infty}|u|^2 dt.$$

On the other hand, for $x=\exp t$ condition (58) takes the form

$$\limsup_{\rho\to\infty}\rho\int_{\rho}^{\infty}|\tilde{q}(t)+\alpha_n^2|\,dt=\infty.$$

Therefore, for any finite B_n and, in particular, for the value B_n determined in the following by formula (59), there exists for $\alpha>0$ a sequence $x_k\to\infty$, for which

$$\lim_{k\to\infty}x_k\int_{x_k}^{\infty}|\tilde{q}(t)+\alpha_n^2|\,dt>B_n^2+\alpha.$$

To prove the theorem it is sufficient to construct for any N an N-finite function $\varphi(t)$, for which we have

$$F[\varphi]=\int_{N'}^{\infty}|\varphi^{(n)}(t)|^2 dt+\sum_{j=1}^{n-1}(-1)^{n-j}\beta_{2j}\int_{N'}^{\infty}|\varphi^{(n-j)}(t)|^2 dt+$$

$$+\int_{N'}^{\infty}\{\tilde{q}(t)+\alpha_n^2\}|\varphi(t)|^2 dt<0.$$

As in the proof of the previous theorem, we determine the function $\varphi(t)$ by formulas (56), replacing x in them by t.
Setting

$$-\beta_{2n-2}\int_0^1|\omega'(z)|^2 dz=B_n^2, \tag{59}$$

we have

$$F[\varphi]\leqslant\frac{1}{\rho}\left\{\frac{B_n^2}{1-\mu}+\rho\sum_{j=0}^{n-2}(-1)^{n-j}\beta_{2j}\left[\frac{1}{[\rho(1-\mu)]^{2(n-j)-1}}+\right.\right.$$

$$\left.+\frac{1}{[R(\nu-1)]^{2(n-j)-1}}\right]\cdot\int_0^1|\omega^{(n-j)}(z)|^2 dz+\frac{\rho B_n^2}{R(\nu-1)}-$$

$$\left.-\rho\int_{\rho}^{R}|\tilde{q}(t)+\alpha_n^2|\,dt\right\}.$$

Choosing the parameters μ, ρ, R and ν in a way similar to the choice of these parameters in the proof of the previous theorem, we obtain

$$F[\varphi]<0,$$

Q. E. D.

In the above proof, a less restrictive condition than (58) was used, namely, that the left-hand side of (58) should exceed the value of B_n^2, determined by (59).

In contrast with the previous theorems, the criterion for oscillatory behavior given below is not connected with an assumption on the negativity of the potential.

Theorem 13 /31 (9)/. If

$$\int_0^\infty q(x)\,dx = -\infty, \tag{60}$$

then equation (36) is oscillatory.

Proof. Using the function $y(x)$ constructed according to formulas (56), we obtain

$$\Phi[y] = \frac{A_n^2}{\rho^{2n-1}(1-\mu)^{2n-1}} + \frac{A_n^2}{R^{2n-1}(\nu-1)^{2n-1}} + \\ + \int_{\mu\rho}^{\rho} S_\rho^2(x)\,q(x)\,dx + \int_\rho^R q(x)\,dx + \int_R^{\nu R} S_R^2(x)\,q(x)\,dx,$$

where

$$S_\rho(x) = \omega(z) \quad \text{for} \quad z = \frac{x-\mu\rho}{\rho(1-\mu)},$$

$$S_R(x) = \omega(z) \quad \text{for} \quad z = \frac{\nu R - x}{R(\nu-1)}.$$

For $R < x < \nu R$ we obviously have the inequalities

$$S_R(x) > 0, \quad S_R'(x) < 0. \tag{61}$$

We set in (56) $\mu < 1$ ρ, $\nu > 1$. This is permissible since these are arbitrary positive parameters. We choose the number R_0 so that for $R > R_0$ we have

$$\frac{A_n^2}{\rho^{2n-1}(1-\mu)^{2n-1}} + \frac{A_n^2}{R^{2n-1}(\nu-1)^{2n-1}} + \\ + \int_{\mu\rho}^{\rho} S_\rho^2(x)\,q(x)\,dx + \int_\rho^R q(x)\,dx < 0. \tag{62}$$

Furthermore we put

$$Q(x) = \int_{R_0}^x q(x)\,dx,$$

so that by virtue of (60) we have $\lim_{x\to\infty} Q(x) = -\infty$.

Let $R_1 \geqslant R_0$ denote the last root of the function $Q(x)$, so that for $x > R_1$ we have $Q(x) < 0$.

Setting $R = R_1$ in (56) and using inequalities (61), we obtain

$$\int_{R_1}^{\nu R_1} S_{R_1}^2(x)\,q(x)\,dx = \int_{R_1}^{\nu R_1} S_{R_1}^2(x)\,Q'(x)\,dx = S_{R_1}^2(x)\,Q(x)\Big|_{R_1}^{\nu R_1} - \\ - 2\int_{R_1}^{\nu R_1} S_{R_1}(x)\,S_{R_1}'(x)\,Q(x)\,dx = \\ = -2\int_{R_1}^{\nu R_1} S_{R_1}(x)\,S_{R_1}'(x)\,Q(x)\,dx < 0.$$

which in combination with (62) gives the required inequality $\Phi[y] < 0$.

For $n = 1$ and $q(x) \geqslant 0$ Theorem 13 gives the criterion established by A. Winter /23 (5)/. We also note that for $n = 1$ it is easy, by iterating transformation (42), to arrive at the following refinement, due to E. Hille (see also /10/), of criterion (40) for oscillatory behavior.

Theorem 14. Let the inequality

$$q(x) < -\frac{1}{4x^2} - \frac{1}{4x^2 \ln^2 x} - \ldots - \frac{1+\delta}{4x^2 \ln^2 x \ldots \ln_r^2 x} \tag{63}$$

hold for large x and some $\delta > 0$ and some natural number r, where $\ln_k = \ln \ln_{k-1}$. Then equation (36) is oscillatory for $n = 1$.

The next two theorems, following from Theorems 8 and 11, give the necessary and sufficient conditions for h-stable non-oscillatory and oscillatory behavior.

Theorem 15 /13 (5)/. For equation (36) to be h-stable non-oscillatory it is sufficient, and in the case when $q(x) \leqslant 0$ also necessary, that the following limit relation be satisfied

$$\lim_{x \to \infty} x^{2n-1} \int_x^\infty |q^*(t)|\, dt = 0. \tag{64}$$

Proof. From condition (64) it follows for any $h > 0$ that

$$\limsup_{x \to \infty} x^{2n-1} \int_x^\infty \frac{1}{h} |q^*(t)|\, dt < \frac{a_n^2}{2n-1}. \tag{65}$$

Thus, condition (37) is satisfied for equation (2), which is consequently non-oscillatory by virtue of Theorem 8.

Suppose it is given that $q(x) \leqslant 0$, and equation (36) is h-stable non-oscillatory. If, regardless of (64), we had

$$\limsup_{x \to \infty} x^{2n-1} \int_x^\infty |q(t)|\, dt = \gamma > 0,$$

then for some $h = h_0$ the following inequality would be satisfied

$$\limsup_{x \to \infty} x^{2n-1} \int_0^\infty \frac{1}{h} |q(t)|\, dt > A_n^2. \tag{66}$$

This coincides with condition (53) for equation (2), which consequently would be oscillatory for $h = h_0$ by virtue of Theorem 11. This contradicts the h-stable non-oscillatory character of equation (36).

Theorem 16 /13 (5)/. For equation (36) to be h-stable oscillatory, it is necessary and, in the case when $q(x) \leqslant 0$ also sufficient, that the following limit relation be satisfied

$$\limsup_{x \to \infty} x^{2n-1} \int_x^\infty |q^*(t)|\, dt = \infty. \tag{67}$$

Proof. Suppose equation (36) is h-stable oscillatory. If condition (66) is not satisfied, then for some value $h = h_0$ inequality (65) holds; therefore,

on the basis of Theorem 8, it follows that equation (2) is non-oscillatory for $h = h_0$. This contradicts the h-stable oscillatory behavior of equation (36).

Conversely, if $q(x) \leqslant 0$ and condition (67) is satisfied, then by the same token for any $h > 0$, inequality (66) which is equivalent to condition (53) for equation (2), is satisfied. Consequently from Theorem 11, the equation is oscillatory.

The concept of the h-stable non-oscillatory and oscillatory behavior was introduced for the first time by Z. Nehari /74/. In that paper oscillatory or non-oscillatory behavior is called strong, and Theorem 16 is established for the case $n = 1$ in a different way. An earlier proof of the strong oscillatory behavior of equation (36) under condition (67) for the case $n = 1$ is due to E. Hille /105/.

31. The positive part of the continuous spectrum. The criteria given below are based on the comparison of operation (1) with the operation

$$(-1)^n \frac{d^{2n}}{dx^{2n}}. \tag{68}$$

The continuous part of the spectrum of this operation covers the semi-axis $\lambda \geqslant 0$.

Theorem 17 /31 (2)/. If

$$\lim_{x \to \infty} \sup |q(x)| = M < \infty, \tag{69}$$

then any interval of the semi-axis $\lambda \geqslant 0$ of length greater than $2M$ contains points of the continuous part of the spectrum of the operator $\tilde{L}$.

Proof. Let us assume that $\lambda_0 > 0$ and $\delta > M$, but that the interval $[\lambda_0 - \delta, \lambda_0 + \delta]$ does not contain points of the set $C(\tilde{L})$. By virtue of the principle of decomposition we have

$$C(\tilde{L}) = C(\tilde{L}_\gamma),$$

and consequently, for any $\gamma \geqslant 0$, we have

$$C(\tilde{L}_\gamma) \cap [\lambda_0 - \delta_1, \lambda_0 + \delta] = \emptyset. \tag{70}$$

Now let the decomposition point γ be chosen so that for $x > \gamma$ we have $|q(x)| \leqslant \delta_0 < \delta$. We then conclude from (70) by means of Theorem 7 of No. 3, that the point λ_0 does not belong to the continuous part of the spectrum of the operator, defined by operation (68) on the semi-axis $x \geqslant \gamma$. Such a conclusion is absurd, since the continuous part of the spectrum of this operator covers the entire semi-axis $\lambda \geqslant 0$.

In /97 (3)/ it was shown for the first time for $n = 1$ that under condition (69) the set $C(\tilde{L})$ is not empty.

Setting $M = 0$ in (69), we obtain the following result, which also follows directly from Theorem 23 of No. 8.

Theorem 18 /31 (2)/. If

$$\lim_{x \to \infty} q(x) = 0,$$

then the entire semi-axis $\lambda \geqslant 0$ is covered by the continuous part of the spectrum of the operator $\tilde{L}$.

Theorem 18 was originally established for $n = 1$ in /101/.

Denoting

$$\sigma = \lim_{x \to \infty} \inf q(x), \quad \rho = \lim_{x \to \infty} \sup q(x)$$

and replacing the potential $q(x)$ by the function $q(x)-\frac{1}{2}(\sigma+\rho)$, we may deduce from Theorem 17 a more precise result.

Theorem 19 /31 (2)/. If

$$\omega=\lim_{x\to\infty}\sup q(x)-\lim_{x\to\infty}\inf q(x), \tag{71}$$

then any interval of the semi-axis $\lambda>\frac{1}{2}(\sigma+\rho)$ *of length larger than* ω *contains points of the continuous part of the spectrum of the operator* $\tilde{L}$.

For $n=1$ Theorem 19 was established differently in /79 (2)/. This theorem means that the lengths of the gaps, which may be formed in the continuous part of the spectrum of operation (68) when it is perturbed by the potential $q(x)$, do not exceed the oscillation of this potential at infinity. The problem of the gaps in the continuous part of the spectrum is considered in greater detail in Chapter V for the case when $n=1$.

Theorem 19 may be refined in that it is possible to consider the point x in (71) as belonging to some set which constitutes a part of the semi-axis $x\geqslant 0$. For this refinement and for the proof of some subsequent theorems, use is made of a system of test functions obtainable by means of the following construction.

Let $\lambda\geqslant 0$ and $\eta>0$ be arbitrarily given. We seek a $2n$-fold continuously differentiable finite function $\varphi(x;\lambda,\eta)$ satisfying the inequality

$$\|(-1)^n\varphi^{(2n)}-\lambda\varphi\|<\eta\|\varphi\|.$$

Such a function may be found by means of the propositions of No. 9, since the continuous part of the spectrum of operation (68) covers the semi-axis $\lambda\geqslant 0$. We also define

$$\varphi_k(x)\equiv\varphi_k(x;\lambda,\eta)=\varphi(x-\alpha_k;\lambda,\eta)\quad(k=1,2,\ldots). \tag{72}$$

where the numbers α_k are chosen so that the supports of the finite functions $\varphi_j(x)$ and $\varphi_k(x)$ do not intersect for $j\neq k$. Taking the linear envelope of the sequence $\{\varphi_k(x)\}$, we obtain an infinite dimensional linear manifold on which the following inequality is satisfied

$$\|(-1)^n\varphi^{(2n)}-\lambda\varphi\|<\eta\|\varphi\|.$$

In particular, we may assume for $\varepsilon>0$

$$\varphi(x;\lambda,\eta)=\sqrt{\varepsilon}\,\varphi_0(\varepsilon x)\exp\left(ix\sqrt[2n]{\lambda}\right), \tag{73}$$

where $\varphi_0(x)$ is an arbitrarily assigned $2n$-fold continuously differentiable finite function with a support $|x|\leqslant 1$, whose norm and maximum of the modulus are equal to 1. It is obvious that for a fixed $\lambda\geqslant 0$, the value of η will tend to zero together with ε.

In the function $\varphi(x;\lambda,\eta)$ in (72) is determined by formula (73), then for all natural numbers k, we have

$$\|\varphi_k(x;\lambda,\eta)\|=1$$

and

$$\max_{0\leqslant x<\infty}|\varphi_k(x;\lambda,\eta)|=\sqrt{\varepsilon}.$$

Using Theorem 9 of No. 3, and the system of test functions (72), we obtain the following theorem which contains the results of Theorems 17, 18 and 19.

Theorem 20 /31 (3)/. If J' is a set of points of the semi-axis $x \geqslant 0$, containing intervals of any length and

$$\omega = \lim_{x\to\infty} \sup q(x) - \lim_{x\to\infty} \inf q(x) \qquad (x \in J'), \tag{74}$$

then for

$$\lambda > \lim_{x\to\infty} \inf q(x) \qquad (x \in J')$$

the length of the gap in $C(\tilde{L})$ does not exceed ω. The middle of the gap is at the point λ.

The following three theorems give integral criteria for the continuity of the positive part of the spectrum.

Theorem 21 /31 (3)/. If the following inequality is satisfied for any $\delta > 0$

$$\int_{M_\delta} |q(x)|^2 dx < \infty, \tag{75}$$

where M_δ is a set of values of x for which $|q(x)| > \delta$, then the positive part of the spectrum of the operator $\tilde{L}$ covers the semi-axis $\lambda \geqslant 0$.

Proof. Using the system of test functions, defined by formulas (72) and (73), we obtain for any $\lambda \geqslant 0$

$$\begin{aligned} \|\tilde{L}\varphi_k - \lambda\varphi_k\| \leqslant \eta\|\varphi_k\| + \|q\varphi_k\| \leqslant \eta\|\varphi_k\| + \\ + \left[\int_{J-M_\delta} |q(x)\varphi_k(x)|^2 dx\right]^{1/2} + \\ + \left[\int_{M_\delta} |q(x)\varphi_j(x)|^2 dx\right]^{1/2} \leqslant \rho(\varepsilon, \delta)\|\varphi_k\|, \end{aligned} \tag{76}$$

where

$$J = [0, \infty), \quad \rho(\varepsilon, \delta) = \eta + \delta + \sqrt{\varepsilon}\left\{\int_{M_\delta} |q(x)|^2 dx\right\}^{1/2}.$$

From inequality (76) and Theorem 9 of No. 3, it follows that

$$C(\tilde{L}) \cap [\lambda - \rho(\varepsilon, \delta), \lambda + \rho(\varepsilon, \delta)] \neq \emptyset,$$

and since the value of $\rho(\varepsilon, \delta)$ can be made arbitrarily small together with ε and δ, then $\lambda \in C(\tilde{L})$, Q. E. D.

It can easily be seen that (75) implies (11). Condition (75) ensures therefore that the relation $C(\tilde{L}) = [0, \infty)$ holds. It is also obvious that in the statement of Theorem 21, as in (74), we may consider that $x \in J'$.

Theorem 22. If for some sequence of intervals Δ_k, whose length $|\Delta_k|$ increases without-bound, the limit relation

$$\lim_{k\to\infty} \frac{1}{|\Delta_k|} \int_{\Delta_k} |q(x)|^2 dx = 0$$

is satisfied, then the positive part of the spectrum of the operator $\tilde{L}$ covers the entire semi-axis $\lambda \geqslant 0$.

Proof. Putting in (73)

$$\varepsilon = \varepsilon_k = \frac{1}{2|\Delta_k|}$$

and taking for the points α_k in (72) the middles of the intervals Δ_k, we obtain the normalized sequence of functions

$$\varphi_k(x) = \varphi(x - \alpha_k;\ \lambda,\ \eta),$$

for which we have

$$\|\tilde{L}\varphi_k - \lambda\varphi_k\| \leqslant \eta_k \|\varphi_k\| + \|q\varphi_k\| \leqslant$$
$$\leqslant \eta_k + \left\{ \frac{1}{2|\Delta_k|} \int_{\Delta_k} |q(x)|^2\, dx \right\}^{1/2},$$

so that

$$\lim_{k\to\infty} \|\tilde{L}\varphi_k - \lambda\varphi_k\| = 0,$$

therefore $\lambda \in C(\tilde{L})$.

The following theorem requires the use of the test system

$$\varphi_k(x;\ \lambda,\ \varepsilon) = \sqrt{\varepsilon}\, \varphi_0(\varepsilon[x - \alpha_k])\, e_k(x;\ \lambda), \tag{77}$$

where $e_k(x;\ \lambda)$ is the solution of the equation

$$(-1)^n y^{(2n)} + q(x)\, y = \lambda y,$$

satisfying the following conditions for $x = \alpha_k$

$$\frac{d^r}{dx^r} e_k(x;\ \lambda)\Big|_{x=\alpha_k} = \frac{d^r}{dx^r} \exp\left\{ i \sqrt[2n]{\lambda}\,(x - \alpha_k) \right\}\Big|_{x=\alpha_k}$$
$$(r = 0,\ 1,\ \ldots,\ 2n-1).$$

We first prove the following simple lemma.

Lemma 1. If the sequence of functions $q_m(x)$ satisfies for some $a < \infty$ the condition

$$\lim_{m\to\infty} \int_0^a |q_m(x)|\, dx = 0,$$

then the solution $\varphi_m(x;\ \lambda)$ of the equation

$$(-1)^n y^{(2n)} + q_m(x)\, y = \lambda y, \tag{78}$$

satisfying the initial conditions

$$\frac{d^r}{dx^r} \varphi_m(x;\ \lambda)\Big|_{x=0} = \frac{d^r}{dx^r} \exp\left(i \sqrt[2n]{\lambda}\, x \right)\Big|_{x=0} \tag{79}$$
$$(r = 0,\ 1,\ \ldots,\ 2n-1),$$

is given, in the sense of uniform convergence in the interval $[0,\ a]$, by

$$\lim_{m\to\infty} \varphi_m^{(r)}(x;\ \lambda) = \frac{d^r}{dx^r} \exp\left(i \sqrt[2n]{\lambda}\, x \right) \quad (r = 0,\ 1,\ \ldots,\ 2n-1). \tag{80}$$

Proof. From (78) and (79) it follows that the function $\varphi_m(x;\ \lambda)$ satisfies Volterra's integral equation

$$\varphi_m(x;\ \lambda) = \exp\left(i \sqrt[2n]{\lambda}\, x \right) - \int_0^x K(x,\ s)\, q_m(s)\, \varphi_m(s;\ \lambda)\, ds. \tag{81}$$

where $K(x, s)$ is the Cauchy function for the differential equation (78) with $q_m(x) \equiv 0$. Applying the method of successive approximations to this integral equation, or using the inequality

$$|\varphi_m(x; \lambda)| \leqslant 1 + \int_0^x |K(x, s)||q_m(s)||\varphi_m(s; \lambda)|\, ds$$

and the lemma in /61 (1) (page 54)/, we obtain the estimate

$$|\varphi_m(x; \lambda)| \leqslant \exp\left\{ M \int_0^a |q_m(s)|\, ds \right\},$$

where

$$M = \max_{0 \leqslant s \leqslant x \leqslant a} |K(x, s)|,$$

so that for $m \to \infty$, the functions $\varphi_m(x; \lambda)$ remain uniformly bounded in the interval $0 \leqslant x \leqslant a$. But then (80) follows from (81), and the relations

$$\frac{d^r}{dx^r} \varphi_m(x; \lambda) =$$

$$= \frac{d^r}{dx^r} \exp\left(i \sqrt[2n]{\lambda} x\right) - \int_0^x \frac{\partial^r K(x, s)}{\partial x^r} q_m(s) \varphi_m(s; \lambda)\, ds$$

Q. E. D.

Theorem 23 /31 (3)/. Suppose that for some sequence of intervals Δ_k, whose length $|\Delta_k|$ increases without bound, we have

$$\lim_{k \to \infty} \int_{\Delta_k} |q(x)|\, dx = 0,$$

then the positive part of the spectrum of the operator $\tilde{L}$ covers the semi-axis $\lambda \geqslant 0$.

Proof. Putting in (77)

$$\varepsilon = \varepsilon_k = \frac{1}{2|\Delta_k|}$$

and taking the middles of the intervals a_k for the points Δ_k, we obtain the normalized sequence of functions $\{\varphi_k\}$, for which

$$L\varphi_k - \lambda\varphi_k = \sqrt{\varepsilon} \sum_{\nu=1}^{2n} \binom{2n}{\nu} e^{\nu} \varphi_0^{(\nu)} e_k^{(2n-\nu)}.$$

From this and from inequality (80) we have the following limit relation for any positive λ

$$\lim_{k \to \infty} \|\tilde{L}\varphi_k - \lambda\varphi_k\| = 0.$$

Therefore $\lambda \in C(\tilde{L})$.

When $n = 1$ Theorems 21 and 23 give, in particular, the criteria previously obtained in a different way, i.e., Putnam's $q(x) \in \mathscr{L}_2(0, \infty)$ /79 (1)/. Wallach's $\sqrt{x} q(x) \in \mathscr{L}_2(0, \infty)$ /19 (1)/, and H. Weyl's $q(x) \in \mathscr{L}_1(0, \infty)$ /20 (2)/, see also /61 (1)/).

We note that all the criteria of the present subsection, which ensure that the relation $[0, \infty) \subset C(\tilde{L})$ holds, are preserved when $q(x)$ is replaced by $\frac{1}{h} q(x)$ and, consequently, they ensure the h-stability of this property. Another criterion of this kind is given by the following theorem.

Theorem 24. If for some (and consequently for any) $\omega \neq 0$

$$\lim_{x\to\infty} \int_x^{x+\omega} |q(t)|^2\, dt = 0, \tag{82}$$

then the entire positive λ*-semi-axis belongs to* $C(\tilde{L})$.

Proof. If $\lambda > 0$, then the point λ belongs to the continuous part of the spectrum of operation (68). Consequently there exists in $\mathfrak{D}_L$ a normalized sequence of finite functions $\{\varphi_k(x)\}$ with disjoint supports, for which

$$\lim_{k\to\infty} \|(-1)^n \varphi_k^{(2n)} - \lambda\varphi_k\| = 0. \tag{83}$$

Further, from (83) it follows that the quadratic functional

$$\int_0^\infty (|y^{(n)}|^2 + |y|^2)\, dx$$

is bounded on the sequence $\{\varphi_k\}$ by a number independent of k, that is

$$\int_0^\infty (|\varphi_k^{(n)}|^2 + |\varphi_k|^2)\, dx < C.$$

On the other hand, from (82) and inequality 18 of No. 27, we have for an arbitrarily given $\eta > 0$ with $k > N_\eta$ that

$$\int_0^\infty q^2\varphi_k^2\, dx < 2\eta(1+C_n) \int_0^\infty (|\varphi_k^{(n)}|^2 + |\varphi_k|^2)\, dx < 2C(1+C_n)\eta.$$

Thus we have the inequality

$$\begin{aligned} \|L\varphi_k - \lambda\varphi_k\| &\leqslant \|(-1)^n \varphi_k^{(2n)} - \lambda\varphi_k\| + \|q\varphi_k\| \leqslant \\ &\leqslant \|(-1)^n \varphi_k^{(2n)} - \lambda\varphi_k\| + \sqrt{2C(1+C_n)\eta}. \end{aligned}$$

and therefore

$$\lim_{k\to\infty} \|L\varphi_k - \lambda\varphi_k\| = 0.$$

Consequently $\lambda \in C(\tilde{L})$, Q. E. D.

The following compactness criterion, due to M. Sh. Birman, has essentially been found in the proof of Theorems 3 and 4.

Lemma 2 /13 (5)/. Let the operator A *in* $\mathfrak{L}_2(0, \infty)$ *be defined on* $\dot{C}_{2n}(0, \infty)$ *by the equality*

$$Ay = (-1)^n y^{(2n)} + y. \tag{84}$$

A necessary and sufficient condition for any A*-bounded set of functions of* $\mathfrak{D}_A$ *to be compact in* $\mathfrak{L}_\rho^2(0, \infty)$ *is that for some (and consequently for any)* $\omega > 0$ *we have*

$$\lim_{x\to\infty} \int_x^{x+\omega} \rho(t)\, dt = 0. \tag{85}$$

Proof. If for some $\omega > 0$ we had for some numerical sequence $x_k \to \infty$

$$\int_{x_k}^{x_k+\omega} \rho(x)\,dx > \rho > 0,$$

then (see the proof of Theorem 4) there would exist in $\mathfrak{D}_A$ an infinite A-normalized sequence of finite functions having disjoint supports and satisfying the inequality

$$\int_0^\infty \rho(x)|\varphi_k(x)|^2\,dx > \rho.$$

But this contradicts the compactness in $\mathscr{L}^2_\rho(0,\infty)$ of the A-normalized sequence $\{\varphi_k\}$, since for the latter, by virtue of Hausdorff's theorem on the ε-net, given any $\varepsilon > 0$ and the corresponding $N = N_\varepsilon$, the expression

$$\int_N^\infty \rho(x)|\varphi_k(x)|^2\,dx < \varepsilon$$

should hold for all k.

The necessity of condition (85) is thus proved. To prove the sufficiency of this condition we conclude from it, as in the proof of Theorem 3, that there exists for any $\varepsilon > 0$ a number $N = N_\varepsilon$ such that for any function $y \in \mathfrak{D}_A$ the inequality

$$\int_N^\infty \rho(x)|y|^2\,dx \leqslant \varepsilon \int_N^\infty (|y^{(n)}|^2 + |y|^2)\,dx \tag{86}$$

is satisfied.

Now if $\mathfrak{M}$ is some A-bounded set of functions of $\mathfrak{D}_A$, then for all $y \in \mathfrak{M}$ we have

$$\int_0^\infty (|y^{(n)}|^2 + |y|^2)\,dx < C \tag{87}$$

and, a fortiori

$$\int_0^\infty |y^{(n)}|^2\,dx < C. \tag{88}$$

The compactness of $\mathfrak{M}$, in the sense of convergence in $\mathscr{L}^2_\rho(0, N)$, follows from (88) Furthermore, using inequalities (86) and (87) and the compactness of in $\mathscr{L}^2_\rho(0, N)$, we construct, as in the proof of Theorem 25 of No. 8, a finite ε-net. From these relations and the construction it follows that $\mathfrak{M}$ is compact in $\mathscr{L}^2_\rho(0, \infty)$.

In the proof of the following theorem of M. Sh. Birman, only the sufficiency of the above-proved criterion for the complete continuity of the inclusion operator H_A in $\mathscr{L}^2_\rho(0, \infty)$ is used.

Theorem 25 /13 (5)/. If for some (and consequently for any) $\omega > 0$

$$\lim_{x\to\infty} \int_x^{x+\omega} |q(t)|\,dt = 0, \tag{89}$$

then the continuous part of the spectrum of the operator $\tilde{L}$ coincides with the semi-axis $\lambda \geqslant 0$.

Proof is based on the general theorems on relatively complete continuity. From Lemma 2 and Theorem 16 of No. 4 it follows that the operator representing multiplication by $|q(x)|$ is completely continuous relative to the positive definite operator (84). But then, from Theorem 17 of No. 4, the multiplication operator is also completely continuous relative to A. Therefore, on the basis of Theorem 19 of No. 5, the continuous parts of the spectra of operation (84) and

$$(-1)^n y^{(2n)} + y + q(x)\, y$$

coincide. The theorem is therefore proved.

Criterion (89), established by Theorem 25, is in a certain sense also necessary for the equality

$$C(\tilde{L}) = [0, \infty). \tag{90}$$

In other words, if for some h-stable semi-bounded operator $\tilde{L}$ with a negative potential $q(x) \leqslant 0$ the property (90) is h-stable, then condition (89) is satisfied. This follows directly from Theorem 4, since the negative part of the spectrum of such an operator is clearly h-stable bounded and discrete.

32. The spectral alternative for semi-bounded Schroedinger operators. In studying the character of the spectrum of the operator $\tilde{L}$, we encountered no case where the non-empty set $C(\tilde{L})$ was bounded from above. In this connection the question arises whether it is permissible to state that either the set $C(\tilde{L})$ is empty, or it is not bounded from above (see /23 (31)/. For $n = 1$, that is, for the Schroedinger operation

$$-y'' + q(x)\, y, \tag{91}$$

this alternative was first established by C. Putnam /79 (4)/ for $q(x) \geqslant 0$. However it is correct for all cases when the potential $q(x)$ is bounded from below. The present subsection is devoted to the proof of two theorems of C. Putnam. These theorems have a more general form than that given by Putnam, since the requirement of the non-negativeness of the potential is replaced by the less restrictive condition of h-stable positiveness of the operator L defined by operation (91).

Theorem 26 /79 (4)/. If the operator L defined by operation (91) is h-stable positive, then the continuous part of its spectrum is either absent, or is not bounded from above. In the latter case, the length of the gap in $C(L)$ with a center at the point λ does not exceed $O(\sqrt{\lambda})$ for $\lambda \to \infty$.

Proof. If $C(L) \neq \emptyset$, then some non-negative $\lambda < a$ belongs to $C(L)$ and according to No. 9, there exists in $\mathfrak{D}_L$ a normalized sequence of finite functions $\{\varphi_k(x)\}$ with disjoint supports such that

$$\|L\varphi_k\| \leqslant a. \tag{92}$$

By virtue of the h-stable positiveness of the operator L we then have for $0 < h < 1$

$$(1-h)\|\varphi_k'\|^2 < \int_0^\infty (|\varphi_k'|^2 + q|\varphi_k|^2)\, dx = (L\varphi_k, \varphi_k) \leqslant a. \tag{93}$$

Setting

$$\psi_k(x) = \varphi_k(x) \cos \sqrt{\lambda} x,$$

we obtain

$$L\psi_k - \lambda\psi_k = L[\varphi_k]\cos\sqrt{\lambda}x + 2\sqrt{\lambda}\varphi_k'(x)\sin\sqrt{\lambda}x,$$

so that

$$\|L\psi_k - \lambda\psi_k\| \leqslant \|L\varphi_k\| + 2\sqrt{\lambda}\|\varphi_k'\|.$$

From the above together with (93) we have the following inequality:

$$\|L\psi_k - \lambda\psi_k\| < a + \sqrt{\frac{a\lambda}{1-h}}. \tag{94}$$

Now, in order to obtain information about the gap length in $C(L)$ from inequality (94), it is necessary to find a lower bound of the norms of the function ψ_k. The functions φ_k may then be regarded as real, since if this were not so we could replace φ_k by $\operatorname{Re}\varphi_k$ or $\operatorname{Im}\varphi_k$, choosing from the two that variant in which the norm of the new test function would be less than $1/\sqrt{2}$.

We have

$$\|\psi_k\|^2 = \int_0^\infty |\varphi_k(x)|^2\cos^2\sqrt{\lambda}x\,dx =$$

$$= \frac{1}{2}\|\varphi_k\|^2 + \frac{1}{2}\int_0^\infty |\varphi_k(x)|^2\cos 2\sqrt{\lambda}x\,dx. \tag{95}$$

but

$$\int_0^\infty |\varphi_k(x)|^2\cos 2\sqrt{\lambda}x\,dx = -\frac{1}{\sqrt{\lambda}}\int_0^\infty \varphi_k(x)\varphi_k'(x)\sin 2\sqrt{\lambda}x\,dx.$$

Consequently,

$$\left|\int_0^\infty \varphi_k^2(x)\cos 2\sqrt{\lambda}x\,dx\right| \leqslant$$

$$\leqslant \frac{1}{\sqrt{\lambda}}\int_0^\infty |\varphi_k(x)\varphi_k'(x)|\,dx \leqslant \frac{1}{\sqrt{\lambda}}\|\varphi_k\|\|\varphi_k'\|,$$

or from (93)

$$\left|\int_0^\infty \varphi_k^2(x)\cos 2\sqrt{\lambda}x\,dx\right| \leqslant \sqrt{\frac{a}{\lambda(1-h)}}. \tag{96}$$

From (95) and (96) it follows that

$$\|\psi_k\|^2 > \frac{1}{2} - \frac{1}{2}\sqrt{\frac{a}{\lambda(1-h)}}. \tag{97}$$

From inequality (94) we have the following relation for $\lambda \to \infty$:

$$\|L\psi_k - \lambda\psi_k\| \leqslant O(\sqrt{\lambda})\|\psi_k\|.$$

This shows that the distance of the points λ from $C(L)$ does not exceed $O(\sqrt{\lambda})$. Thus it follows in particular that the set $C(L)$ is not bounded from above.

The following theorem shows that when the point $\lambda = 0$ belongs to the continuous part of the spectrum of an h-stably positive Schroedinger operator, the continuous part of the spectrum of this operator covers the entire semi-axis $\lambda \geqslant 0$ without gaps.

Theorem 27 /79 (4)/. If the operator L, defined by operation (91) is h-stably positive, then either the point $\lambda=0$ does not belong to the continuous parts of its spectrum, or the continuous part of the spectrum, $C(L)$, covers the entire semi-axis $\lambda \geqslant 0$.

Proof. If the point $\lambda=0$ belongs to $C(L)$, then for some normalized sequence of finite functions $\varphi_k(x) \in \mathfrak{D}_L$ with disjoint supports we have

$$\lim_{k\to\infty} L\varphi_k = 0,$$

that is, instead of inequality (92) we have for $k \to \infty$

$$\|L\varphi_k\| = o(1).$$

But then, replacing the number a by $o(1)$ in (94) and (97), we obtain, for an arbitrary fixed $\lambda > 0$ when $k \to \infty$, the relation

$$\|L\psi_k - \lambda\psi_k\| = o(1),$$

thus it follows that the point λ belongs to the set $C(L)$.

It is obvious that both theorems of the present subsection remain valid, if the operator defined by operation (91) on the semi-axis $x \geqslant N$ is h-stably positive for some $N > 0$. In particular, it is sufficient to assume for the validity of these theorems that

$$\liminf_{x\to\infty} q(x) \geqslant 0.$$

Using the criterion of A. M. Molchanov, R.S. Ismagilov extended the spectral alternative, expressed by theorem 26, to the case of the two-term operation (1) of any order $2n$ with a potential $q(x)$ bounded from below. $O(\sqrt{\lambda})$ is then replaced by $O\left(\sqrt[2n]{\lambda}\right)$.

The corresponding proof may be given as follows. If $C(L) \neq \theta$, then according to Theorem 6 of A. M. Molchanov, there exists an infinite sequence of disjoint intervals ω_k of equal length, for which

$$\int_{\omega_k} q(x)\,dx < C. \tag{98}$$

Here $C < \infty$ and we may assume that $q(x) \geqslant 1$. It is required to show that for sufficiently large λ we have $C(L) \cap [\lambda, \lambda + O(\sqrt{\lambda})] \neq \theta$. For this it is sufficient to find in $\mathfrak{D}_L$ an infinite dimensional manifold G, on which we have

$$\|l[\varphi] - \lambda\varphi\| < \delta\|\varphi\| \tag{99}$$

when $\delta = O(\sqrt{\lambda})$.

By virtue of well-known asymptotic formulas, there exists a number $\mu > 0$, such that for $\lambda > \mu$ in any interval $[\lambda, \lambda + O(\sqrt{\lambda})]$ there are eigenvalues of operation (91) on the interval ω_1 with zero boundary conditions. Clearly, we may assume that there are several such eigenvalues, so that from the eigenfunctions corresponding to them it is possible to form a linear combination contained in $\mathfrak{D}_L$. Since inequality (99) is satisfied for each of these eigenfunctions, it is also satisfied by their linear combination $\varphi_1 \in \mathfrak{D}_L$. Furthermore, it can be shown that due to condition (98), the value of μ, from which the "asymptotic behavior" begins, is independent of k. In particular, this was essentially established in /67 (1)/ for $n = 2$. For $n > 2$ one has to follow the calculations in /82 (2)/ or /72 (3)/. Constructing the function $\varphi_k \in \mathfrak{D}_L$ in each interval ω_k for $k \geqslant 2$, we obtain inequality (99) on the infinite dimensional envelope of finite functions $\varphi_k(x)$ with disjoint supports.

The spectral alternative for operation (1) is thus established for any n.

33. The case when the spectrum of the Schroedinger operator covers the entire axis. In this case the potential should obviously be unbounded from below. But since unboundedness from below of the potential does

not imply unboundedness from below of the operator L (see, for example, Theorem 2 of No. 27), we subject the potential to a more restrictive condition by assuming that

$$\lim_{x\to\infty} q(x) = -\infty.$$

The spectrum $S(\tilde{L})$ is then unbounded not only from above but also from below. This can easily be shown by considering the corresponding quadratic functional in the case of an infinite system of finite functions with disjoint supports, formed by a displacement from a fixed finite function belonging to $\mathfrak{D}_L$.

E. Titchmarsh and D. Sears have established /88/ that if some additional conditions are satisfied in the case $n=1$ the spectrum $S(L)$ covers the entire λ-axis, provided the potential tends to $-\infty$ sufficiently slowly and is discrete. For the opposite case (see also /61 (1)/. This theorem was generalized by I. M. Rapoport to differential operations of any order /82 (2)/ by means of asymptotic methods.

In the present subsection we establish a spectrum continuity criterion close to the above-mentioned criterion of Titchmarsh and Sears.

Theorem 28. Let the differentiable function $q(x)$ tend monotonically to $-\infty$ for $x\to\infty$, preserving the direction of convexity for large x. If the potential tends to $-\infty$ so slowly that

$$\lim_{x\to\infty} \frac{q'(x)}{\sqrt{|q(x)|}} = 0, \tag{100}$$

then the spectrum of the Schroedinger operator covers the entire λ-axis without gaps.

Proof. We first assume that the function $q(x)$ is twice continuously differentiable, and raising operation (91) to the second power we obtain

$$(L^2y,\ y) = \int_0^\infty \{|y''|^2 + 2q|y'|^2 + (q^2 - q'')|y|^2\}\,dx.$$

If λ_0 is an arbitrary real number, then, to prove that it belongs to $C(L)$, it is sufficient, according to the results of No. 10, to show that for an arbitrarily small $\alpha > 0$ and for any $N > 0$ there exists a N-finite function $y \in \mathfrak{D}_{L^2}$, for which

$$\Phi[y] = \int_0^\infty \{|y''|^2 - 2\rho|y'|^2 + (\rho^2 + \rho'' - \alpha)|y|^2\}\,dx < 0, \tag{101}$$

where $\rho(x) = \lambda_0 - q(x)$. Clearly we then have $\lim_{x\to\infty} \rho(x) = \infty$, and for some $\gamma > 0$ we have for all $x > \gamma$ that

$$\rho(x) > 0,\quad \rho(x) \geqslant 0,\quad \rho''(x) \geqslant 0 \quad (\text{or } \leqslant 0)$$

and

$$\rho'(x) = o\left(\sqrt{\rho(x)}\right).$$

Let us assume that for some natural number m, and $n > m$ we have

$$y(x) = \begin{cases} 0 & (t < m\pi), \\ \left(1 - \frac{t}{n\pi}\right)\sin t & (m\pi \leqslant t \leqslant n\pi), \\ 0 & (t > n\pi), \end{cases} \tag{102}$$

where

$$t = \int_{\gamma}^{x} \sqrt{\rho(x)}\, dx.$$

If the function $y(x)$ is smoothed at the point $t = m\pi$, so that at $t = m\pi$ its first derivative is continuous, then according to Lemma 5 of No. 10, it is sufficient to show inequality (10) for the function (102). Postponing for the time being the smoothing, we obtain, on substituting the function (102) in the left-hand side of inequality (101)

$$\Phi[y] = \int_{0}^{\infty} f(t;\ n)\, dt. \tag{103}$$

where

$$f(t;\ n) = \rho^{3/2}\left[\frac{2}{n^2\pi^2}\cos 2t + \frac{2}{n^2\pi^2}\cos^2 t + \right.$$
$$+ \frac{4}{n\pi}\left(1 - \frac{t}{n\pi}\right)\sin 2t - 2\left(1 - \frac{t}{n\pi}\right)^2 \cos 2t\Big] +$$
$$+ \frac{\dot{\rho}^2}{4\sqrt{\rho}}\left[\frac{1}{n^2\pi^2}\sin^2 t - \frac{1}{n\pi}\left(1 - \frac{t}{n\pi}\right)\sin 2t + \left(1 - \frac{t}{n\pi}\right)^2 \cos^2 t\right] +$$
$$+ \dot{\rho}\sqrt{\rho}\left[\frac{1}{n^2\pi^2}\sin 2t - \frac{1}{n\pi}\left(1 - \frac{t}{n\pi}\right)\cos 2t - \left(1 - \frac{t}{n\pi}\right)\frac{1}{n\pi}\cos^2 t - \right.$$
$$- \frac{1}{2}\left(1 - \frac{t}{n\pi}\right)^2 \sin 2t - \frac{2}{n\pi}\left(1 - \frac{t}{n\pi}\right)\sin^2 t + 2\left(1 - \frac{t}{n\pi}\right)^2 \sin t\Big] -$$
$$- \frac{\alpha}{\sqrt{\rho}}\left(1 - \frac{t}{n\pi}\right)^2 \sin^2 t.$$

From the conditions of the theorem, the function $\rho[t] = \rho(x)$ has the following properties: $\dot{\rho}\sqrt{\rho}$ does not decrease (or does not increase), $\dot{\rho} \geqslant 0$, $\dot{\rho}(t) = o(1)$ and $\rho[t] = o(t)$ for $t \to \infty$. Using these properties in estimating the integral (103), the following inequality is easily obtained by means of integration by parts and by the second theorem on the mean for a fixed m:

$$\Phi[y] = o(\sqrt{n}) - \alpha \int_{m\pi}^{n\pi} \frac{1}{\sqrt{\rho}}\left(1 - \frac{t}{n\pi}\right)^2 \sin^2 t\, dt.$$

The integral on the right-hand side of this relation is for $n \to \infty$ positive, and tends to infinity faster that $\sqrt{n}$, so that for sufficiently large n we have

$$\Phi[y] < 0.$$

This inequality obviously remains valid when the function (102) is first smoothed by eliminating the discontinuity of the first derivative at the point $t = m\pi$, and then taking n sufficiently large.

We have yet to dispense with the additional assumption on the smoothness of the potential. From Theorem 23 of No. 8, it is sufficient to show that for any function $q(x)$ satisfying the conditions of Theorem 28, a twice continuously differentiable function $\tilde{q}(x)$ may be constructed which satisfies all the conditions of Theorem 28 as well as the condition

$$\lim_{x \to \infty} \{q(x) - \tilde{q}(x)\} = 0.$$

To construct the function $\tilde{q}(x)$ we first choose a positive function $\varepsilon(x)$, tending to zero for $x \to \infty$, having the derivative $\varepsilon'(x) = o[\rho'(x)]$. Since

condition (100) holds for the function $q(x)$, it also holds for any function $\tilde{q}(x)$ satisfying the inequalities

$$|q(x)-\tilde{q}(x)|<\varepsilon(x), \quad |q'(x)-\tilde{q}'(x)|<\varepsilon'(x). \tag{104}$$

If the potential $q(x)$ satisfies the conditions of theorem 28, then it can easily be seen that among the functions $q(x)$ satisfying inequalities (104), there exist twice continuously differentiable functions which for $x\to-\infty$ tend monotonically to $-\infty$, while preserving the direction of the convexity. This completes the proof of Theorem 28.

From the above theorem, it follows in particular that if the potential $q(x)$, which preserves the direction of the convexity, tends monotonically to $-\infty$ more slowly than $-x^2$, then the continuous part of the spectrum $S(L)$ of the operator covers the entire λ-axis. On the other hand, it follows from the theorem of Titchmarsh and Sears that if the potential tends to $-\infty$ in the same way, but at a higher rate than $-x^2$, the spectrum $S(\tilde{L})$ of the operator $\tilde{L}$ is then discrete.

34. Examples of the study of the character of the spectrum in the case of a one-term operation. The methods described are applicable also to the one-term differential operation

$$(-1)^n\frac{d^n}{dx^n}\left(p\frac{d^ny}{dx^n}\right) \qquad (x\geqslant 0,\ p(x)>0). \tag{105}$$

Let $n=1$, so that operation (105) takes the form

$$\mu[y]=-\frac{d}{dx}p\frac{dy}{dx}.$$

The self-adjoint operators generated by operation (105) will be denoted by $\tilde{L}$.

We now wish to isolate a complete square from operation $\mu[y]$. To this end we introduce the first-order operation

$$\nu[y]=i\rho\frac{d}{dx}\rho, \tag{106}$$

where $\rho(x)>0$ for $x\geqslant 0$. We continue $\rho(x)$ symmetrically to the entire axis, and denote by P the minimum closed symmetrical operator, generated by the operation $\mu[y]$ in $\mathfrak{L}_2(-\infty,\infty)$. Since the general solution of the equation $P^*z-\lambda z=0$ is

$$z=\frac{C}{\rho}\exp\left\{-i\lambda\int_0^x\frac{dx}{\rho^2}\right\},$$

then for $\frac{1}{\rho(x)}\in\mathfrak{L}_2(0,\infty)$ both defect numbers of the operator P are equal to one, and for $\frac{1}{\rho(x)}\in\mathfrak{L}_2(0,\infty)$ the operator P is self-adjoint.

Let the operator, unitary in $\mathfrak{L}_2(-\infty,\infty)$, representing multiplication by the function

$$\exp\left\{-i\lambda\int_0^x\frac{dx}{\rho^2}\right\},$$

be denoted by U_λ, where $\operatorname{Im}\lambda=0$.
We have for any finite function $y\in\mathfrak{D}_L$

$$U_\lambda^*PU_\lambda y=(P+\lambda I)\,y,$$

so that

$$U_\lambda^* P U_\lambda = P + \lambda I.$$

If the operator P is self-adjoint, it follows from the last relation that its real spectrum, being a non-empty set, should fill the entire λ-axis.

Assuming the function $p(x)$ sufficiently smooth, and separating from $\mu[y]$ the complete square of operation (106), we obtain

$$-\frac{d}{dx} p \frac{dy}{dx} = -\rho \frac{d}{dx} \rho^2 \frac{d}{dx} \rho y + \frac{1}{3} (\rho^3)'' \rho y,$$

that is,

$$\mu[y] = v^2[y] + \frac{1}{3} (\rho^3)'' \rho y,$$

where $\rho = \sqrt[4]{p}$.

If

$$\lim_{x \to \infty} \rho (\rho^3)'' = +\infty, \tag{107}$$

then (see the proof of Theorem 5 of No. 28) the spectrum of the operator $\tilde{L}$ is semi-bounded from below and discrete, and if there exists a finite limit

$$\lim_{x \to \infty} \rho (\rho^3)'' = \beta, \tag{108}$$

then, from Theorem 23 of No. 8, we have

$$C(\tilde{L}) = C\left(P^2 + \frac{\beta}{3} I\right).$$

Suppose, for instance, it is required to investigate the character of the spectrum of the operator $\tilde{L}_\alpha$, generated in $\mathscr{L}_2(0, \infty)$ by the operation $-\frac{d}{dx}(1+x)^\alpha \frac{dy}{dx}$, so that $\rho(x) = (1+x)^{\alpha/4}$.

For $\alpha \leqslant 2$ we have $\frac{1}{\rho} \in \mathscr{L}_2(0, \infty)$, so that $P = P^*$ and, according to the above, the set $S(P)$ covers the entire λ-axis. Since the limit relation (108) also holds, then

$$C(\tilde{L}_\alpha) = \begin{cases} [0, \infty) & \text{for } \alpha < 2, \\ \left[\frac{1}{4}, \infty\right) & \text{for } \alpha = 2. \end{cases}$$

For $\alpha > 2$ condition (107) is satisfied, so that the spectrum of the operator $\tilde{L}_\alpha$ is semi-bounded from below and discrete.

By making the substitution

$$\xi := \int_0^x \frac{dx}{p(x)}$$

the spectral problem for the operation $\mu[y]$ is transformed into the spectral problem for the polar operation of the type (29). In this manner the problem of the criterion for the discreteness of the spectrum of the operator $\tilde{L}$ for $n = 1$ reduces to the same problem for the polar operation (29) on a finite interval. For this operation M. Sh. Birman obtained /13 (8)/ a criterion similar to relation (31). This criterion, as applied to the operation $\mu[y]$, is expressed by the relation

$$\lim_{x \to \infty} x \int_x^\infty \frac{dx}{p(x)} = 0.$$

For $n > 1$, the passage to the polar operation is impossible, and the problem arises of the direct derivation of the condition for the discreteness of the spectrum. This problem was studied by V. A. Tkachenko, who showed that for the spectrum of the operator $\tilde{L}$ to be discrete in the general case (105), it is sufficient that the limit relation

$$\lim_{x\to\infty} x^{2n-1}\int_x^\infty \frac{dx}{p(x)} = 0 \tag{109}$$

is satisfied.

V. A. Tkachenko's proof reduces to the following.

First the following two inequalities are established for any finite function $y \in \mathfrak{D}_L$

$$\int_0^\infty \frac{|y^{(n-1)}y^{(n)}|\,dx}{\int_x^\infty \frac{dt}{p(t)}} \leqslant 2\int_0^\infty p y^{(n)2}\,dx$$

and

$$\int_0^\infty y^2\,dx \leqslant C\int_0^\infty |y^{(n-1)}y^{(n)}|\,x^{2n-1}\,dx.$$

The constant C depends only on n.

Obviously, it is sufficient to establish the first of these inequalities for $n = 1$. It then follows from the Cauchy-Bunyakovskii inequality that

$$\int_0^\infty \frac{|yy'|\,dx}{\int_x^\infty \frac{dt}{p(t)}} \leqslant \left\{\int_0^\infty \frac{y^2\,dx}{p(x)\left[\int_x^\infty \frac{dt}{p(t)}\right]^2}\right\}^{1/2}\left\{\int_0^\infty p y'^2\,dx\right\}^{1/2}$$

and from this relation that

$$\begin{aligned}\frac{1}{2}\int_0^\infty \frac{y^2\,dx}{p(x)\left[\int_x^\infty \frac{dt}{p(t)}\right]^2} &= \left|\int_0^\infty \frac{dx}{p(x)\left[\int_x^\infty \frac{dt}{p(t)}\right]^2}\int_x^\infty yy'\,ds\right| = \\ &= \left|\int_0^\infty yy'\,ds\int_0^s \frac{dx}{p(x)\left[\int_x^\infty \frac{dt}{p(t)}\right]^2}\right| = \\ &= \left|\int_0^\infty yy'\left[\frac{1}{\int_s^\infty \frac{dt}{p(t)}} - \frac{1}{\int_0^\infty \frac{dt}{p(t)}}\right]ds\right| \leqslant \int_0^\infty \frac{|yy'|\,ds}{\int_s^\infty \frac{dt}{p(t)}}.\end{aligned}$$

To prove the second inequality we set $\alpha = 2k-2$ and replace y by $y^{(k-1)}$ in (9). Thus, we obtain the recurrence relation

$$\int_0^\infty x^{2k-2}[y^{(k-1)}]^2\,dx \leqslant \frac{4}{(2k-1)^2}\int_0^\infty x^{2k}[y^{(k)}]^2\,dx,$$

leading to the inequality

$$\int_0^\infty y^2\,dx \leqslant \frac{4^{n-1}}{\prod_{k=0}^{n-1}(2k-1)^2}\int_0^\infty x^{2n-2}\,[y^{(n-1)}]^2\,dx$$

or

$$\int_0^\infty y^2\,dx \leqslant \frac{-2\cdot 4^{n-1}}{\prod_{k=0}^{n-1}(2k-1)^2}\int_0^\infty x^{2n-2}\,dx\int_x^\infty y^{(n-1)}y^{(n)}\,dt \leqslant$$

$$\leqslant \frac{2\cdot 4^{n-1}}{\prod_{k=0}^{n-1}(2k-1)^2}\int_0^\infty |y^{(n-1)}y^{(n)}|\,\frac{x^{2n-1}\,dx}{2n-1},$$

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Now, assuming that condition (109) is satisfied, we find, given an $\varepsilon > 0$, a number N such that for $x \geqslant N$ we have $x^{2n-1}\int_x^\infty \frac{dt}{p(t)} < \frac{\varepsilon}{2C}$. Then, by means of these inequalities we obtain the relation

$$\int_N^\infty y^2\,dx \leqslant C\int_N^\infty \frac{|y^{(n-1)}|\,|y^{(n)}|\,x^{2n-1}\int_x^\infty \frac{dt}{p(t)}}{\int_x^\infty \frac{dt}{p(t)}} \leqslant$$

$$\leqslant \frac{\varepsilon}{2}\int_N^\infty \frac{|y^{(n-1)}y^{(n)}|\,dx}{\int_x^\infty \frac{dt}{p(t)}} \leqslant \varepsilon\int_N^\infty p\,|y^{(n)}|^2\,dx.$$

Hence, according to the decomposition principle it follows that the spectrum of the operator $\tilde{L}$ is discrete.

If the function $p(x)$ is monotonic, then relation (109) is necessary for the discreteness of the spectrum, but in the general case the problem of the necessity of condition (109) for $n > 1$ remains open.

35. On the spectrum of the Schroedinger operator **with complex-valued potential energy.** In the present subsection we give some examples of the application of the decomposition method to the differential operation

$$-y'' + q(x)\,y \qquad (-\infty < x < \infty) \tag{110}$$

with a complex-valued potential. In $\mathfrak{L}_2(-\infty, \infty)$ this operation generates some J-symmetric operator L with the minimal domain of definition. Its J-self-adjoint extensions will be denoted by $\tilde{L}$.

When generalizing some of the previously established propositions to the case of a complex-valued potential we shall confine ourselves to non-integral criteria.

Let Q_N denote the convex envelope of the values of the function $q(x)$ for $|x| \geqslant N$, and let Q_N^+ denote the set of points of the λ-plane formed from Q_N by associating a ray $z+t$ $(0 < t < \infty)$ to each point $z \in Q_N$. The constructed

set Q_N^+ is obviously also convex. It contains all the values of the functional

$$(Ly, y) = \int\limits_{|x|>N} |y'|^2 dx + \int\limits_{|x|>N} q(x)|y|^2 dx. \tag{111}$$

For $y \in \mathfrak{D}_L$ we have $\|y\| = 1$, since the second term in (111) belongs to Q_N, and the first is positive. Denoting the orthogonal sum of the minimal operators, defined by operation (110) in $\mathscr{L}_2(-\infty, 0)$ and $\mathscr{L}_2(0, \infty)$, by L_N, we then find that the above inclusion relation can be written in the form

$$K(L_N) \subset Q_N^+ \tag{112}$$

where we have returned to the notation of No. 21.

Let Q_∞ denote the intersection of all the Q_N and let Q_∞^+ denote the intersection of all the Q_N^+ for $N \to \infty$. The limit set Q_∞^+ is not always obtained from Q_∞ by associating the rays $z + t$ $(z \in Q_\infty)$ to Q_∞. Thus for example in the case of a real potential tending to $-\infty$ we obviously have $Q_\infty = 0$ while the set Q_∞^+ coincides with the real λ-axis.

We note that since the set Q_∞^+ is the intersection of the sets Q_N^+, it is convex and closed. The complement R_∞ of the set Q_∞^+ to the whole λ-half-plane is therefore either empty, or is an open set consisting of no more than two connected components.

Extensive use is made below of the general theorems on linear equations in normed spaces, which have been mentioned previously in § 4 of Chapter I. From these general theorems we have, in particular, the following theorem.

Theorem 29. The spectrum of the operator $\tilde{L}$ has the following properties in the complement R_∞ of the set Q_∞^+ to the whole λ-plane:

1) The spectrum of the operator $\tilde{L}$ in R_∞ has no points other than its eigenvalues.

2) Either the set $S(\tilde{L})$ has no finite limit points inside R_∞, or each point of at least one of the connected components of the domain R_∞ is an eigenvalue of the operator $\tilde{L}$.

3) If a given connected component of the domain R_∞ is not wholly covered by the discrete part of the spectrum of the operator $\tilde{L}$, then the rank of each eigenvalue of the operator $\tilde{L}$, lying in this component, is finite and the corresponding root subspace G_λ is separable according to the formula

$$\mathscr{L}_2(-\infty, \infty) = G_\lambda \dotplus F_\lambda, \tag{113}$$

where F_λ is the orthogonal complement of the subspace JG_λ to the whole $\mathscr{L}_2(-\infty, \infty)$.

Proof is obtained by means of Theorem 45 of No. 23, providing we first establish the finiteness of the characteristic of the operator $\tilde{L}$, and the closure of the manifold $(\tilde{L} - \lambda I)\mathfrak{D}_{\tilde{L}}$ for $\lambda \in R_\infty$. However the finiteness of the characteristic is obvious, and the closure of the manifold $(\tilde{L} - \lambda I)\mathfrak{D}_{\tilde{L}}$ can easily be established as follows.

Let $\lambda \in R_\infty$, that is $\lambda \notin Q_\infty^+$. Then, for a certain N we shall have $\lambda \notin Q_N^+$, thus from (112) we conclude that

$$\lambda \notin K(L_N). \tag{114}$$

But by virtue of Theorem 42 of No. 21, it follows from relation (114) that the point λ belongs to the regularity field $\Pi(L_N)$. Also from the decomposition principle for finite dimensional extensions it follows that $C(\tilde{L}) = C(L_N)$.

Thus $\lambda \in \Pi(L_N)$, and

$$C(\tilde{L}) \cap \Pi(L_N) = C(L_N) \cap \Pi(L_N) = 0,$$

so that $\lambda \notin C(\tilde{L})$, therefore the manifold $(\tilde{L} - \lambda I)\mathfrak{D}_{\tilde{L}}$ is closed. The theorem is thus proved.

The following theorem gives a criterion sufficient for the discreteness of the spectrum of the operator $\tilde{L}$.

Theorem 30. If for sufficiently large $|x|$ the values of the potential $q(x)$ lie inside some angle

$$\alpha < \arg q(x) < \alpha + \pi - \delta \qquad (\delta > 0), \tag{115}$$

which does not contain the negative semi-axis, and

$$\lim_{|x| \to \infty} |q(x)| = \infty, \tag{116}$$

then the spectrum of the operator $\tilde{L}$ is discrete. Neglecting unusual extensions, where the whole λ-plane is covered by the eigenvalues of the operator $\tilde{L}$, the spectrum $S(\tilde{L})$ has no finite limit points, the rank of each eigenvalue is finite and the corresponding root subspaces are separated according to formula (113).

An exceptional case can not occur if the real part of the potential is bounded from below.

Proof. The validity of all these statements apart from the last one, follows directly from the previous theorem, since when conditions (115) and (116) are satisfied the set Q_∞^+ is empty. The last statement follows from Theorem 46 of No. 24 by virtue of which the operator L is J-self-adjoint. As condition (115) shows that the set $K(L)$ does not fill the whole plane, then, a fortiori, the spectrum $S(L) \subset K(L)$ does not cover the whole λ-plane.

For real potentials, the cases when $q(x) \to +\infty$ and $q(x) \to -\infty$ are essentially different. In the first case the spectrum is bounded from below and discrete (Theorem 5), whereas in the second case it may cover the entire λ-axis (Theorem 28). Theorem 30 shows that the second case is exceptional since if the limit relation (116) holds for a constant $\arg q(x) = \alpha$, then for any α with the possible exception of $\alpha = \pi$ the spectrum of the operator $\tilde{L}$ is discrete.

Theorem 30 leads, in particular, to the discreteness criterion established earlier by other methods /63 (1)/ and expressed by the limit relations

$$\lim_{|x| \to \infty} \operatorname{Re} q(x) = +\infty \quad \text{or} \quad \lim_{|x| \to \infty} \operatorname{Im} q(x) = +\infty \ (\text{or } -\infty).$$

Theorem 31. If the minimal differential operator L_1, defined by the operation

$$-y'' + y \operatorname{Re} q(x) \qquad (-\infty < x < \infty)$$

is self-adjoint and

$$\lim_{|x| \to \infty} \operatorname{Im} q(x) = 0, \tag{117}$$

then the operator L is J-self-adjoint and its spectrum has the following properties:

1) $C(L) = C(L_1)$.
2) *Outside $C(L_1)$ the spectrum of the operator L has no finite limit points.*
3) *Eeach eigenvalue of the operator L not belonging to $C(L_1)$ has a finite*

rank and the corresponding root subspace is separable according to Formula (113).

Proof. The J-self-adjointness of the operator L follows from the boundedness of the imaginary part of the potential, and statement 1) follows directly from the obvious generalization to complex-valued potentials of Theorem 23 of No. 8 on small perturbations (see No. 24).

Furthermore, it follows from condition (117) that the set Q_∞^+ lies on the real λ-axis, and therefore each of the half-planes $\operatorname{Im}\lambda > 0$ and $\operatorname{Im}\lambda < 0$ belongs to a connected component of the complements R_∞ of the set Q_∞^+. But then, by virtue of Theorem 30, in the presence of at least one non-real finite limit point of the spectrum $S(L)$, the corresponding half-plane should have belonged wholly to the discrete part of the spectrum of the operator L, which is impossible since outside the strip

$$\inf \operatorname{Im} q(x) \leqslant \operatorname{Im}\lambda \leqslant \sup \operatorname{Im} q(x)$$

the operator L cannot have eigenvalues.

Statement (2) is thus also proved. The last statement follows directly from Theorem 29.

From Theorem 31 we obtain the following proposition which is a generalization of Theorems 1 and 18 to the case of complex-valued potentials.

Theorem 32. If

$$\lim_{|x|\to\infty} q(x) = 0,$$

then the operator L is J-self-adjoint, and its continuous spectrum $C(L)$ coincides with the semi-axis $\lambda \geqslant 0$. Outside this region the spectrum $S(L)$ has only isolated eigenvalues of finite rank, their corresponding root subspaces being separable according to formula (113).

Theorems 31 and 32 were essentially established by other methods due to V. B. Lidskii /63 (2)/. A. M. Molchanov has generalized his discreteness criterion to complex-valued potentials (see No. 59).

36. The spectrum of two-term differential operators on vector functions. Most of the results of Nos. 27—32 can easily be carried over to the operators $\tilde{L}$ generated in $\mathfrak{L}_2(0, \infty)$ by the vector operation

$$l[y] = (-1)^n y^{(2n)} + Q(x)y \qquad (0 \leqslant x < \infty), \tag{118}$$

where $Q(x) = (q_{jk}(x))$ is a Hermitian matrix of the m-th order and

$$y(x) = \{y_k(x)\}_{k=1}^m \qquad (m < \infty).$$

However, actually carrying out the appropriate proofs is superfluous, since it would reduce to a repetition of the corresponding calculations of Nos. 27—32.

We confine ourselves only to the necessary modifications in the formulation of the theorems when passing from the scalar operation (1) to the vector operation (118). Let $\mu(x)$ and $\nu(x)$ denote, respectively, the smallest and largest eigenvalues of the matrix $Q(x)$.

The quadratic functional corresponding to operation (118) is of the form

$$\Phi[y] = \int_0^\infty \sum_{k=1}^m |y_k^{(n)}|^2 dx + \int_0^\infty \sum_{j,k=1}^m q_{jk} y_j \bar{y}_k \, dx.$$

This permits the minimum and maximum limit estimates

$$\|y^{(n)}\|^2+\int_0^\infty \mu(x)\sum_{k=1}^m |y_k(x)|^2\,dx \leqslant \Phi[y] \leqslant$$

$$\leqslant \|y^{(n)}\|^2+\int_0^\infty \nu(x)\sum_{k=1}^m |y_k(x)|^2\,dx.$$

The previous criteria for semi-boundedness from below and discreteness of the entire spectrum or of its negative part may be generalized to the case of the vector operation (118). However, we must then replace $q(x)$ by $\mu(x)$. This is also true for the criteria for non-oscillatory behavior (Theorems 1, 2, 5, 8, 9, 10). When generalizing the criteria for oscillatory behavior (Theorems 9, 11—14), $q(x)$ should be replaced by $\nu(x)$. We note that although the left-hand sides of inequalities (39) and (40) are now different in the formulation of Knezer's generalized theorem, the accuracy of the corresponding criterion for the vector operation does not decrease. In other words, in terms of the eigenvalues of the matrix $Q(x)$, the criteria defined by Theorem 9 cannot be substantially improved, since for $\mu(x)<-(\alpha_n^2+\delta)x^{-2n}$ or $\nu(x)\geqslant-\alpha_n^2x^{-2n}$ operation (118) may be both non-oscillatory and oscillatory.

When extending the theorems on the positive part of the continuous spectrum (Theorems 17—22) to the vector case, one has to replace lim inf $q(x)$, lim sup $q(x)$ and $|q(x)|$ by lim inf $\mu(x)$, lim sup $\nu(x)$ and the Euclidian norm $\|Q(x)\|$ of the matrix $Q(x)$ respectively.

The formulation of the spectral alternatives of No. 32 does not need to be modified for $n=1$.

Considerable modification is required in the formulation of Theorem 28, for the case when the spectrum covers the entire λ-axis. The reason for this is connected with the fact that, contrary to the scalar case, the smallest eigenvalue of the matrix $Q''(x)$ not only does not coincide, but is in no way connected to $\nu''(x)$. The problem of extending the theorems for the necessary and sufficient conditions for some arrangement of the parts of the spectrum to the case of operation (118) remains open.

37. A discrete analog of the oscillation theorems. According to Theorem 32 of No. 14, a necessary and sufficient condition for the solutions of the finite-difference equation

$$-\alpha_k y_{k-1}-\alpha_{k+1}y_{k+1}+\beta_k y_k=\lambda y_k \qquad (119)$$

$$(\alpha_k>0,\ k=0,\ 1,\ \ldots)$$

to have an oscillatory character, is that for $\lambda=\lambda_0$ the part of the spectrum of the Jacobian matrix

$$t=\begin{pmatrix}\beta_0 & \alpha_1 & 0 & \cdot\\ \alpha_1 & \beta_1 & \alpha_2 & \cdot\\ 0 & \alpha_2 & \beta_2 & \cdot\\ \cdot & \cdot & \cdot & \cdot\end{pmatrix} \qquad (120)$$

lying to the left of the point $\lambda=\lambda_0$ be an infinite set. Thus, as in No. 12, the investigation of the oscillatory character of equation (119) is connected with the corresponding quadratic functional

$$\Phi[y]=(t[y],\ y)-\lambda_0(y,\ y) \qquad (121)$$

on N-finite vectors (that is, on finite vectors whose first N coordinates are equal to zero).

The proposition corresponding to Theorem 28 of No. 10 may be stated as follows: if for some N given any N-finite vector y we have $\Phi[y] \geqslant 0$, then there is only a finite set of points of the spectrum of the operator $\tilde{L}$ to the left of the point λ_0; if for each N there exists an N-finite vector y for which $\Phi[y] < 0$, then there is an infinite set of points of the spectrum of the operator $\tilde{L}$ to the left of the point λ_0.

In accordance with (121) the functional $\Phi[y]$ has the form

$$\Phi[y] = \alpha_N |y_N|^2 + \alpha_{N+p} |y_{N+p}|^2 + \\ + \sum_{k=N+1}^{N+p} \alpha_k |y_{k-1} + y_k|^2 + \sum_{k=N}^{N+p} \omega_k |y_k|^2, \tag{122}$$

where

$$\omega_k = -\alpha_k - \alpha_{k+1} + \beta_k - \lambda_0. \tag{123}$$

The two-term differential operation (1) corresponds to the Jacobian matrix (120) with $\alpha_k \equiv 1$. The following proposition is the discrete analog of Theorem 13.

Theorem 33. If

$$\sum_{k=0}^{\infty} (\beta_k - 2) = -\infty,$$

then the equation

$$-y_{k+1} - y_{k-1} + \beta_k y_k = 0$$

is oscillatory.

Proof. Suppose that for any given N we have in (122)

$$y_k = \begin{cases} 0, & k < N, \\ (-1)^k, & N \leqslant k \leqslant N+p, \\ 0, & k > N+p. \end{cases}$$

We then obtain for a sufficiently large natural number p the inequality

$$\Phi[y] = 2 + \sum_{k=N}^{N+p} (\beta_k - 2) < 0.$$

We now give the discrete analogue of Knezer's theorem /71 (1)/. We assume that there exists a finite positive limit

$$\lim_{k \to \infty} \alpha_k = \alpha > 0$$

and that for $k \to \infty$ with some $s > 0$ the following asymptotic equality holds

$$\alpha_k - \alpha = Ck^{-s} + O(k^{-s-1}). \tag{124}$$

Here C is a real constant of an arbitrary sign. In particular, relation (124) is satisfied for $\alpha_k \equiv 1$ (providing we set $C = 0$).

Theorem 34 /71 (1)/. If relation (124) is satisfied, then the equation

$$-\alpha_k y_{k-1} - \alpha_{k+1} y_{k+1} + \beta_k y_k = 0$$

is non-oscillatory for

$$\lim_{k\to\infty} \inf k^2\omega_k > -\frac{1}{4} \tag{125}$$

and oscillatory for

$$\lim_{k\to\infty} \sup k^2\omega_k < -\frac{1}{4}. \tag{126}$$

Proof. Suppose relations (124) and (125) are satisfied. We have for any natural numbers N and p and a real number y_k that

$$\sum_{N+1}^{N+p} \alpha_k \left(y_k + y_{k-1} - \frac{y_k - y_{k-1}}{4k}\right)^2 = \sum_{N+1}^{N+p} \alpha_k (y_k + y_{k-1})^2 + \\ + \frac{1}{16} \sum_{N+1}^{N+p} \alpha_k \frac{(y_k - y_{k-1})^2}{k^2} - \frac{1}{2} \sum_{N+1}^{N+p} \alpha_k \frac{y_k^2 - y_{k-1}^2}{k}.$$

Hence, using the elementary inequality

$$2\,|y_k y_{k-1}| \leqslant y_{k-1}^2 + y_k^2,$$

we obtain

$$0 < \sum_{N+1}^{N+p} \alpha_k \left(y_k + y_{k-1} - \frac{y_k - y_{k-1}}{4k}\right)^2 \leqslant \sum_{N+1}^{N+p} \alpha_k (y_k + y_{k-1})^2 - \\ - \frac{1}{2} \sum_{N+1}^{N+p} \frac{\alpha_k y_k^2}{k} + \frac{1}{2} \sum_{N}^{N+p-1} \frac{\alpha_{k+1} y_k^2}{k+1} + \frac{1}{16} \left\{ \sum_{N+1}^{N+p} \alpha_k \frac{y_k^2}{k^2} + \right. \\ \left. + \sum_{N}^{N+p-1} \alpha_{k+1} \frac{y_k^2}{(k+1)^2} + \sum_{N+1}^{N+p} \alpha_k \frac{y_k^2}{k^2} + \sum_{N+1}^{N+p} \alpha_k \frac{y_{k-1}^2}{k^2} \right\} \leqslant \\ \leqslant \sum_{N+1}^{N+p} \alpha_k (y_k + y_{k-1})^2 + \frac{1}{2} \sum_{N}^{N+p} \left(\frac{\alpha_{k+1}}{k+1} - \frac{\alpha_k}{k} + \right. \\ \left. + \frac{\alpha_k}{4k^2} + \frac{\alpha_{k+1}}{4(k+1)^2} \right) y_k^2 + \frac{\alpha_N}{2N} y_N^2 \left(1 - \frac{1}{4N}\right) - \\ - \frac{\alpha_{N+p+1}}{2(N+p+1)} \left(1 - \frac{1}{4(N+p+1)}\right) y_{N+p}^2. \tag{127}$$

In order to compare the right-hand side of inequality (127) with (122), we now prove the following inequality with the aid of (125)

$$\omega_k > \frac{1}{2\alpha} \left\{ \frac{\alpha_k}{4k^2} - \frac{\alpha_k}{k} + \frac{\alpha_{k+1}}{4(k+1)^2} + \frac{\alpha_{k+1}}{k+1} \right\}. \tag{128}$$

For $\alpha_k \equiv \alpha$ inequality (128) is an obvious consequence of condition (125). In the general case the right-hand side of (128) is equal to $-\frac{1}{4k^2} + o\left(\frac{1}{k^2}\right)$ because of (124). Thus inequality (128) again follows from condition (125).

Comparing the right-hand side of inequality (127) with expression (122) for $\Phi[y]$, the required relation $\Phi[y] \geqslant 0$ is obtained from (128) for some N given any N-finite real vector.

Turning to the proof of the second part of the theorem we assume that condition (126) is fulfilled and define the test vector y by the equalities

$$y_k = \begin{cases} 0 & (0 \leqslant k \leqslant N_1), \\ (-1)^k \dfrac{(k - N_1)\sqrt{N_2}}{N_2 - N_1} & (N_1 \leqslant n \leqslant N_2), \\ (-1)^k \sqrt{k} & (N_2 \leqslant k \leqslant N_3), \\ (-1)^k \dfrac{\sqrt{N_3}(N_4 - k)}{(N_4 - N_3)} & (N_3 \leqslant k \leqslant N_4), \\ 0 & (k \geqslant N_4). \end{cases} \tag{129}$$

For the vector y so constructed we have

$$\Phi[y] = \frac{N_2}{(N_2-N_1)^2}\sum_{N_1+1}^{N_2}\{\alpha_k+\alpha\omega_k(k-N_1)^2\}+$$
$$+\sum_{N_2+1}^{N_3}\{\alpha_k(\sqrt{k}-\sqrt{k-1})^2+\alpha k\omega_k\}+$$
$$+\frac{N_3}{(N_4-N_3)^2}\sum_{N_3+1}^{N_4}\{\alpha_k+(N_4-k)^2\omega_k\}. \tag{130}$$

Denoting the terms on the right-hand side of the latter relation by S_1, S_2 and S_3 respectively, we have for S_1 and S_3 the inequalities

$$S_1 < \frac{N_2}{(N_2-N_1)^2}\sum_{N_1+1}^{N_2}\alpha_k,$$
$$S_3 < \frac{N_3}{(N_4-N_3)}\sum_{N_3+1}^{N_4}\alpha_k.$$

For sufficiently large N_1 it may be assumed that $\alpha_k \equiv \alpha$, so that

$$S_1 < \frac{2\alpha N_2}{(N_2-N_1)}, \quad S_3 < \frac{2\alpha N_3}{(N_4-N_3)}$$

or, if the numbers N_j are chosen to satisfy a geometric progression $N_{j+1} = N_j \cdot r$ $(j = 1, 2, 3;\ r > 1)$,

$$S_1 < \frac{2\alpha r}{r-1}, \quad S_3 < \frac{2\alpha}{r-1}. \tag{131}$$

Now, setting $\alpha_k = \alpha + \varepsilon_k$ and using inequality (126) we obtain for S_2 for any $\varepsilon > 0$ the estimate

$$S_2 < \sum_{N_2+1}^{N_3}\left\{\alpha(\sqrt{k}-\sqrt{k-1})^2 - \alpha\frac{1+\varepsilon}{4k} + \varepsilon_k(\sqrt{k}-\sqrt{k-1})^2\right\}$$

or, using the expansion

$$(\sqrt{k}-\sqrt{k-1})^2 = \frac{1}{4k} + O\left(\frac{1}{k^2}\right),$$

we obtain the estimate

$$S_2 < \sum_{N_2+1}^{N_3}\left\{\alpha\left[O\left(\frac{1}{k^2}\right) - \frac{\varepsilon}{4k}\right] + \frac{\varepsilon_k}{4k} + O\left(\frac{1}{k^2}\right)\right\},$$

so that for large r we have

$$S_2 < -\frac{\alpha\varepsilon}{8}\ln r. \tag{132}$$

Now, from (130), (131), and (132) we find the required inequality

$$\Phi[y] < 0,$$

thus the theorem is proved.

Chapter III

THE SPECTRUM OF ONE-DIMENSIONAL DIFFERENTIAL OPERATORS OF GENERAL FORM

38. One-dimensional differential operations of general form. In this chapter we consider some problems concerning the spectrum of the self-adjoint operators $\tilde{L}$ generated, in the sense of No. 7, by the operation

$$l=\sum_{k=0}^{n}(-1)^{n-k}\frac{d^{n-k}}{dx^{n-k}}p_k(x)\frac{d^{n-k}}{dx^{n-k}} \tag{1}$$

in the space $\mathfrak{L}_2(0, \infty)$. The conditions which the coefficients of this operation satisfy are given in No. 7. It is always assumed that $p_0(x)>0$, and in the majority of the subsequent theorems we take $p_0(x)\equiv 1$. In the particular case when $p_0(x)\equiv 1$, $p_k(x)=0$ $(k=1, 2, \ldots, n-1)$, operation (1) reduces to the two-term operation considered in the previous chapter. The criteria given below for the various arrangements of the parts of the spectrum are generalizations of the criteria of Chapter II. However, compared with the previous chapter, the results regarding the spectrum of the operators generated by operation (1) are less complete. In particular, in the present chapter we do not consider necessary criteria, which are much more difficult to obtain than in the previous chapter. There the differential operation was determined only by a single functional parameter, the potential $q(x)$. Several results regarding the spectrum of a multi-term differential operator obtained by asymptotic methods can be found in a monograph by I. M. Rapoport /82 (2)/, and also in a book by M. A. Naimark /72 (3)/.

39. Criteria for the semi-boundedness of the spectrum. The following two theorems are generalizations of Theorems 1 and 2 established in No. 27 for two-term operations.

Theorem 1. If all the coefficients $p_k(x)$ $(k=1, 2, \ldots, n)$ are bounded from below and

$$\liminf_{x\to\infty}p_k(x)=a_k \qquad (k=0, 1, \ldots, n),$$

then to the left of the point

$$\mu=\min_{0\leqslant\lambda<\infty}Q(\lambda),$$

where

$$Q(\lambda)=\sum_{k=0}^{n}a_k\lambda^{n-k},$$

the spectrum of the operator $\tilde{L}$ has only an isolated set of eigenvalues with one possible limit point $\lambda=\mu$.

Proof. Let us choose the decomposition point $\gamma = \gamma_\varepsilon$ so that for $x > \gamma$ we have $p_k(x) > a_k - \varepsilon$ $(k = 0, 1, \ldots, n)$.

If the corresponding decomposition of the operator $\tilde{L}$ is defined by the relations

$$\tilde{L} = \tilde{L}_0, \quad L_0 = L_1 \oplus L_\gamma,$$

then

$$C(\tilde{L}) = C(L_\gamma).$$

But for $y \in \mathfrak{D}_{L_\gamma}$ we have for the operator L_γ

$$(L_\gamma y, y) = \int_\gamma^\infty \sum_{k=0}^n p_k(x) |y^{(n-k)}(x)|^2 dx \geqslant$$

$$\geqslant \int_\gamma^\infty \sum_{k=0}^n (a_k - \varepsilon) |y^{(n-k)}(x)|^2 dx = \tag{2}$$

$$= \left(Q_\varepsilon\left[-\frac{d^2}{dx^2}\right] y, y\right) \geqslant \mu_\varepsilon (y, y),$$

where

$$Q_\varepsilon(\lambda) = \sum_{k=0}^n (a_k - \varepsilon)\lambda^{n-k}$$

and

$$\mu_\varepsilon = \min_{0 \leqslant \lambda < \infty} Q_\varepsilon(\lambda).$$

From inequality (2) it follows that the operator L_γ is semi-bounded, and thus the operator $\tilde{L}$ is also semi-bounded. Furthermore, from inequality (2) it also follows that $C(L_\gamma) \cap (-\infty, \mu_\varepsilon) = \emptyset$, and therefore $C(\tilde{L}) \cap (-\infty, \mu_\varepsilon) = \emptyset$. Taking the limit when $\varepsilon \to 0$ in the latter relation, we obtain

$$C(\tilde{L}) \cap (-\infty, \mu) = \emptyset,$$

Q. E. D.

It follows from this proof that if in Theorem 1 we have $a_n = \infty$, then the spectrum $S(\tilde{L})$ of the operator $\tilde{L}$ is discrete.

In the following theorem, as previously in No. 25, the negative part of a function is denoted by an asterisk.

Theorem 2 /31 (9)/. If $p_0(x) \equiv 1$ and if for any $\delta > 0$

$$\int_{M_{k,\delta}} |p_k^*(x)| dx < \infty \qquad (k = 1, 2, \ldots, n), \tag{3}$$

where $M_{k,\delta}$ is the set of values of x for which $|p_k^*(x)| \leqslant \delta$, then the negative part of the spectrum of the operator $\tilde{L}$ is discrete and bounded.

Proof. For any k $(k = 1, 2, \ldots, n)$ we introduce the quadratic functional

$$\Phi^{(k)}[y] = \int_0^\infty |y^{(n)}| dx + 2n \int_0^\infty p_k^*(x) |y^{(n-k)}|^2 dx + \tag{4}$$

$$+ 2n\eta \int_0^\infty |y^{(n-k)}|^2 dx,$$

which on substitution $y^{(n-k)}(x) = z(x)$ is transformed into a functional of type (12) of No. 27. As was established in the proof of Theorem 2 in No. 27, it follows from conditions (3) that for any system $\eta_k > 0$ $(k = 1, 2, \ldots, n)$

number N can be found such that the functionals $\Phi^{(k)}[y]$ are non-negative on all the N-finite functions of $\mathfrak{D}_{\tilde{L}}$.

Let the quadratic functional

$$\Phi_\varepsilon[y]=\int_0^\infty \sum_{k=0}^{n} p_k(x)\,|y^{(n-k)}(x)|^2\,dx+\varepsilon\int_0^\infty |y(x)|^2\,dx$$

corresponding to operation (1) be represented in the form

$$\Phi_\varepsilon[y]=\int_0^\infty \sum_{k=1}^{n}\left\{\frac{1}{n}\,|y^{(n)}|^2+p_k\,|y^{(n-k)}|^2+\varepsilon_k\,|y^{(n-k)}|^2-\varepsilon_{k-1}\,|y^{(n-k+1)}|^2\right\}dx,$$

where $\varepsilon_0=0$ and $\varepsilon_n=\varepsilon_1$.

From inequality (3) of No. 26 we have

$$\Phi_\varepsilon[y]\geqslant\sum_{k=1}^{n}\left\{\frac{1}{n}\int_0^\infty |y^{(n)}|^2\,dx+\int_0^\infty p_k^*\,|y^{(n-k)}|^2\,dx+\right.$$
$$\left.+\varepsilon_k\int_0^\infty |y^{(n-k)}|^2\,dx-\varepsilon_{k-1}\|y\|^{\frac{k-1}{n}}\|y^{(n)}\|^{\frac{n-k+1}{n}}\right\}. \qquad (5)$$

We choose the small positive numbers ε_k $(k=1,\ 2,\ \ldots,\ n)$ so that $\varepsilon-(\varepsilon_1+\varepsilon_2+\ \ldots\ +\varepsilon_{n-1})$ is positive, and we choose the number N, as in the proof of Theorem 2 of No. 27, so that all the functionals (4) are non-negative on the N-finite functions $y\in\mathfrak{D}_{\tilde{L}}$ with $\eta_k=\varepsilon_k$ $(k=1,\ 2,\ \ldots,\ n-1)$ and $\eta_n=\varepsilon-(\varepsilon_1+\varepsilon_2+\ \ldots\ +\varepsilon_{n-1})$.

Now if y is a normalized N-finite function of $\mathfrak{D}_{\tilde{L}}$, inequality (5) takes the form

$$\Phi_\varepsilon[y]\geqslant\sum_{k=1}^{n}\left\{\frac{1}{n}\int_N^\infty |y^{(n)}|^2\,dx+\int_N^\infty p_k^*\,|y^{(n-k)}|^2\,dx+\right.$$
$$\left.+\varepsilon_k\int_N^\infty |y^{(n-k)}|^2\,dx-\varepsilon_{k-1}\|y^{(n)}\|^{\frac{n-k+1}{n}}\right\}. \qquad (6)$$

We distinguish between two cases: for a given function $y(x)$ we have (a) $\|y^{(n)}\|\geqslant 1$ and (b) $\|y^{(n)}\|\leqslant 1$.

In the first case, replacing $\|y^{(n)}\|^{\frac{n-k+1}{n}}$ by $\|y^{(n)}\|^2$ and taking $\varepsilon_k<\frac{1}{2n}$ $(k=1,\ 2,\ \ldots,\ n-1)$, we obtain from inequality (6)

$$\Phi_\varepsilon[y]\geqslant\sum_{k=1}^{n}\left\{\frac{1}{2n}\int_N^\infty |y^{(n)}|^2\,dx+\int_N^\infty p_k^*\,|y^{(n-k)}|^2\,dx+\right.$$
$$\left.+\varepsilon_k\int_N^\infty |y^{(n-k)}|^2\,dx\right\}=\frac{1}{2n}\sum_{k=1}^{n}\Phi^{(k)}[y]>0.$$

In the second case, replacing $\|y^{(n)}\|^{\frac{n-k+1}{n}}$ by 1, we obtain

$$\Phi_\varepsilon[y]\geqslant\sum_{k=1}^{n-1}\left\{\frac{1}{n}\int_N^\infty |y^{(n)}|^2\,dx+\int_N^\infty p_k^*\,|y^{(n-k)}|^2\,dx+\right.$$
$$\left.+\varepsilon_k\int_N^\infty |y^{(n-k)}|^2\,dx\right\}+\frac{1}{n}\int_N^\infty |y^{(n)}|^2\,dx+\int_N^\infty p_n\,|y|^2\,dx+$$
$$+\left(\varepsilon-\sum_{k=1}^{n-1}\varepsilon_k\right)\int_N^\infty |y|^2\,dx\geqslant\frac{1}{2n}\sum_{k=1}^{n}\Phi^{(k)}[y]\geqslant 0.$$

Thus, in both cases we have $\Phi[y] \geqslant 0$, and the theorem is proved.

In particular, Theorem 2 leads to I. M. Rapoport's criterion /82 (2)/ for the boundedness from below of the operator $\tilde{L}$. This criterion is expressed by the relations

$$\int_0^\infty |p_k(x)|\,dx < \infty \qquad (k = 1, 2, \ldots, n).$$

40. **Domains of non-oscillatory and oscillatory behavior.** As a starting point we consider the differential equation with constant coefficients

$$Q\left[-\frac{d^2}{dx^2}\right]u = 0, \tag{7}$$

where

$$Q(\lambda) = \sum_{k=0}^{n} a_k \lambda^{n-k} \qquad (a_0 = 1). \tag{8}$$

Since the spectrum of the corresponding operator coincides, except for a finite number of eigenvalues, with the semi-axis $\lambda \geqslant \mu$, where

$$\mu = \min_{0 \leqslant \lambda < \infty} Q(\lambda),$$

then (7) is a non-oscillatory equation for $\mu \geqslant 0$ and an oscillatory equation for $\mu < 0$.

The inequality $\mu > 0$ is equivalent to the polynomial $Q(\lambda^2)$ having no real roots.

Let $\mathfrak{R}_a$ denote the set of points $M(a_1, a_2, \ldots, a_n)$ in the n-dimensional space of the coefficients of polynomial (8) to which correspond the non-negative polynomials $Q(\lambda^2)$. It is obvious that $\mathfrak{R}_a$ is the closed convex part of the n-dimensional space containing the "first octant" of the a-system of coordinates.

The set $\mathfrak{R}_a$ can be described analytically by means of Hermite's theorem. According to this theorem an algebraic polynomial has no real roots, if and only if the signature of the quadratic form $\sum_{j,k=0}^{n} S_{j+k}\xi_j\xi_k$, where S_{j+k} is the Newtonian sum of the order $j+k$, is zero. The set $\mathfrak{R}_a$ is thus the closure of the set of points $M(a_1, a_2, \ldots, a_n)$ for which there is exactly n sign changes in the sequence of the $2n$ first principal minors of Hankel's matrix $(S_{j+k})_{j,k=0}^{n}$. The Newtonian sums are calculated successively by means of the well-known recurrent relations

$$ka_k = s_1a_{k-1} + s_2a_{k-2} + \ldots + (-1)^{k-1}s_k,$$

where

$$a_{2k} = a_k, \quad a_{2k+1} = 0 \qquad (k = 0, 1, \ldots, n).$$

By means of substitution (42) of No. 30, the functional $\Phi[u]$ corresponding to equation (7) with constant coefficients takes the form

$$\tilde{\Phi}_0[y] = \int_0^\infty |y^{(n)}|^2\,dx + \sum_{k=1}^{\infty}\int_0^\infty b_k x^{-2k}|y^{(n-k)}|^2\,dx.$$

where the coefficients b_k are obtained from a_k by means of the triangular linear transformation

$$b_k = \Lambda(a_1, a_2, \ldots, a_k) \qquad (k = 1, 2, \ldots, n). \tag{9}$$

Under this linear transformation the convex closed set $\mathfrak{R}_a$ passes into some convex closed set $\mathfrak{R}_b$ in the space of the coefficients b_k $(k = 1, 2, \ldots, n)$. Now, the general theorems of No. 12 lead to the following generalization of Knezer's classical theorem in application to differential operation (1).

Theorem 3 /31 (9)/. Let the convex closed set $\mathfrak{R}_b$ be the transform of the closed convex set $\mathfrak{R}_a$ defined by transformation (9) and let, for $k = 1, 2, \ldots, n$,

$$\hat{b}_k = \lim_{x \to \infty} \sup x^{2k} p_k(x), \quad \check{b}_k = \lim_{x \to \infty} \inf x^{2k} p_k(x).$$

If the point $M(\check{b}_1, \check{b}_2, \ldots, \check{b}_n)$ belongs to $\mathfrak{R}_b$, then

$$\sum_{k=0}^{n} (-1)^{n-k} \frac{d^{n-k}}{dx^{n-k}} p_k(x) \frac{d^{n-k}y}{dx^{n-k}} = 0 \tag{10}$$

is a non-oscillatory equation. If the point $M(\hat{b}_1, \hat{b}_2, \ldots, \hat{b}_n)$ does not belong to $\mathfrak{R}_b$, then it is an oscillatory equation.

In particular, for $n = 1$ the set $\mathfrak{R}_b$ is the semi-axis $b_1 \geqslant -\frac{1}{4}$, for $n = 2$ the set $\mathfrak{R}_b$ is determined by the inequalities $b_2 \geqslant -\frac{9}{4} b_1 - \frac{9}{16}$ $\left(\text{for } b_1 \geqslant -\frac{5}{2}\right)$ and $b_2 \geqslant \frac{1}{4}(2 - b_1)^2$ $\left(\text{for } b_1 \leqslant -\frac{5}{2}\right)$. Thus, for a fourth-order differential equation, the set $\mathfrak{R}_b$ is a part of the plane enclosed between the arc of a parabola and the ray tangent to it (see also /76/). For $n = 3$, the set $\mathfrak{R}_b$ is the convex part of the three-dimensional space containing the first octant of the b-system of coordinates. Here simple calculations lead to the following parametric equations for the boundary of the set $\mathfrak{R}_b$:

$$b_1 = \frac{35}{4} + (u - 2v), \quad b_2 = \frac{259}{16} + 13(u - 2v) + (v^2 - 2uv),$$
$$b_3 = \frac{225}{64} + \frac{225}{16}(u - 2v) + \frac{25}{4}(v^2 - 2uv) + uv^2.$$

We note that in all cases the last of the segments intersected by the set $\mathfrak{R}_b$ on the axes of the b-system coincides with the previously found quantity α_n (see No. 30).

In the classical case $n = 1$, the non-oscillatory character of the equation

$$-y'' + q(x)y = 0, \tag{11}$$

is equivalent to the existence of a solution of the corresponding Riccati equation

$$u' - u^2 - q(x) = 0 \tag{12}$$

on some semi-axis $x \geqslant a$. This follows directly from the form of the transformation

$$u = y'y^{-1},$$

transforming (11) into (12).

In the general case, the non-oscillatory character of equation (10) is also equivalent to the solvability of some non-linear system of differential equations on the semi-axis $x \geqslant a$. To obtain this system, it is sufficient

to make use of Theorem 31 of No. 12 and of a theorem of M. G. Krein (see also No. 44) according to which the non-oscillatory differential operation (1) can have the multiplicative representation

$$l[y] = \mu^* p_0 \mu[y] \qquad (0 < x < \infty). \tag{13}$$

where

$$\mu[y] = y^{(n)} + u_1(x)y^{(n-1)} + \ldots + u_n(x)y$$

is an operation with the coefficients $q(x) \in C_n(0, \infty)$ $(k = 1, 2, \ldots, n)$, and $\mu^*[y]$ is its Lagrange adjoint operation.

Equating the coefficients of the derivatives $y^{(k)}$ $(k = 0, 1, \ldots, 2n-1)$ on both sides of equality (13), we obtain the required system of differential equations for the functions $u_j(x)$ $(j = 1, 2, \ldots, n)$, which for $n = 1$ reduces to the Riccati equation (12).

Equation (10) is non-oscillatory if and only if this non-linear system of differential equations has, for some a, a solution on the semi-axis $x \geqslant a$.

41. On gaps in the continuous part of the spectrum. As was shown in No. 31, the gaps in the continuous part of the spectrum of two-term operations do not exceed the potential oscillations at infinity when $\lambda \to \infty$. In the present section it is shown that under certain conditions, the gaps in the continuous part of the spectrum of general operation (1), cannot increase too rapidly with λ, either.

Theorem 4 /31 (9)/. If $p_0(x) \equiv 1$, and if the other coefficients $p_k(x)$ of differential operation (1) and their derivatives up to order $n-k$ $(k = 1, 2, \ldots, n)$ are bounded on the semi-axis $x \geqslant 0$, then for $\lambda \to \infty$ the length of the gap centered at the point λ does not exceed $O\left(\lambda^{1-\frac{1}{n}}\right)$.

Proof. We denote

$$\omega = \varlimsup_{x \to \infty} p_1(x) - \varliminf_{x \to \infty} p_1(x)$$

and for simplicity first assume that

$$\varlimsup_{x \to \infty} p_1(x) = -\varliminf_{x \to \infty} p_1(x), \tag{14}$$

so that

$$\omega = 2 \varlimsup_{x \to \infty} |p_1(x)|.$$

We write the differential operation (1) in the form

$$l = \sum_{k=0}^{2n} q_k(x) \frac{d^{2n-k}}{dx^{2n-k}}, \tag{15}$$

where $q_0(x) \equiv (-1)^n$; consequently $q_1(x) \equiv 0$.

For any function

$$\varphi_j(x) = \varphi_j(x;\ \lambda, \varepsilon)$$

of the test system (72), (73) of No. 31 we have, for large λ,

$$\|L\varphi_j - \lambda\varphi_j\| \leqslant \|(-1)^n \varphi_j^{(2n)} - \lambda\varphi_j\| + \sum_{k=2}^{2n} \|q_k \varphi_j^{(2n-k)}\| \leqslant$$

$$\leqslant \eta + \left(\varlimsup_{x \to \infty} |p_1(x)| + \frac{\delta}{2}\right)\lambda^{\frac{n-1}{n}} + O\left(\lambda^{\frac{n-2}{n}}\right) \leqslant \eta + \left(\frac{\omega}{2} + \delta\right)\lambda^{\frac{n-1}{n}}, \tag{16}$$

where $\eta > 0$ and $\delta > 0$ are arbitrarily small for sufficiently large λ.

It follows from inequality (16) that the radius $r(\lambda)$ of the gap in $C(\tilde{L})$ centered at the point λ does not exceed $\left(\frac{\omega}{2}+\delta\right)\lambda^{\frac{n-1}{n}}$. Thus we obtain the required relation

$$\limsup_{\lambda\to\infty} \lambda^{\frac{1-n}{n}} r(\lambda) \leqslant \frac{\omega}{2}.$$

In the general case, when equality (14) is not satisfied, the proof should be modified as follows. Setting

$$\tau = \frac{1}{2}\left\{\limsup_{x\to\infty} p_1(x) - \liminf_{x\to\infty} p_1(x)\right\},$$

we represent operation (1) in the form

$$(-1)^n y^{(2n)} + \tau y^{(2n-2)} + [q_2(x) - \tau] y^{(2n-2)} + \sum_{k=3}^{2n} q_k(x) y^{(2n-k)}$$

and in the construction of the test system of functions according to formula (73) of No. 31 we replace the factor $\exp(\iota \sqrt[2n]{\lambda} x)$ by $\exp(\iota\mu x)$, where μ is the only positive root of the equation

$$\mu^{2n} + (-1)^{n-1}\tau\mu^{2n-2} - \lambda = 0,$$

which, for $\lambda \to \infty$, obviously has the form $\sqrt[2n]{\lambda} + o(\sqrt[2n]{\lambda})$.

This proposition is a particular case of the following theorem.

Theorem 5 /31 (9)/. If the coefficients of the differential operation (15) are bounded on the semi-axis $x \geqslant 0$ and

$$\omega_k = \limsup_{x\to\infty} q_k(x) - \liminf_{x\to\infty} q_k(x). \qquad (17)$$

where $\omega_k = 0$ for $k < r$, but $\omega_r \neq 0$, then the upper limit of the ratio of the length of the gap in $C(\tilde{L})$ centered at the point λ to $\omega_r \lambda^{1-\frac{r}{2n}}$ does not exceed unity for $\lambda \to \infty$.

Proof of this theorem reduces to the calculations given in the proof of the previous theorem.

Theorem 5 remains valid, if we take $x \in J'$ in (17) (see No. 31) and assume boundedness of the coefficients of operation (15) on J' only. In the case of a two-term operation, Theorem 5 reduces to Theorem 20 of No. 31.

It follows from the propositions of the present section that the continuous part of the spectrum $C(\tilde{L})$ extends to $+\infty$ whenever the coefficients of operation (15) are bounded on J'.

42. Continuity criteria for the positive part of the spectrum. Since the continuous part of the spectrum of the operator $\tilde{L}$ in the case of the differential operation with constant coefficients $Q\left[-\frac{d^2}{dx^2}\right]$ (see No. 40) covers the semi-axis $\lambda \geqslant \mu$, where $\mu = \min_{\lambda>0} Q(\lambda)$, then for any $\lambda \geqslant \mu$ and $\eta > 0$ a normalized finite function $\psi(x; \lambda, \eta)$ can be found in $\mathfrak{D}_\chi$ which satisfies the inequality

$$\left\| Q\left[-\frac{d^2}{dx^2}\right]\psi - \lambda\psi \right\| < \eta.$$

We construct the test system of functions $\{\psi_j\}_{j=1}^{\infty}$, setting

$$\psi_j(x) \equiv \psi_j(x; \lambda, \eta) = \psi(x - \alpha_j; \lambda, \eta), \tag{18}$$

where the numbers α_j are chosen so that the supports of the finite functions $\psi_j(x)$ and $\psi_k(x)$ are disjoint for $j \neq k$. By means of this system of functions the following theorem may be established.

Theorem 6 /31 (9)/. If $p_0(x) \equiv 1$, and if the coefficients of the differential operation (1) tend "smoothly" to a finite limit, that is,

$$\lim_{x \to \infty} p_k(x) = a_k \tag{19}$$

and

$$\lim_{x \to \infty} p_k^{(r)}(x) = 0 \quad (k = 1, 2, \ldots, n;\ r = 1, 2, \ldots, n-k), \tag{20}$$

then the continuous part of the spectrum $C(\tilde{L})$ covers the semi-axis $\lambda \geqslant \mu$ without gaps.

Proof. For $\lambda > 0$ and the system of functions (18) we have

$$\|L\psi_j - \lambda\psi_j\| \leqslant \left\|L\psi_j - Q\left[-\frac{d^2}{dx^2}\right]\psi_j\right\| + \left\|Q\left[-\frac{d^2}{dx^2}\right]\psi_j - \lambda\psi_j\right\| \leqslant$$

$$\leqslant \left\|\sum_{k=0}^{n}(-1)^{n-k}[p_k(x) - a_k]\frac{d^{2(n-k)}\psi_j}{dx^{2(n-k)}}\right\| +$$

$$+ \left\|\sum_{k=1}^{2n-1} r_k(x)\frac{d^k\psi_j}{dx^k}\right\| + \left\|Q\left[-\frac{d^2}{dx^2}\right]\psi_j - \lambda\psi_j\right\|,$$

therefore, by virtue of (20), the limit relations

$$\lim_{x \to \infty} r_k(x) = 0 \qquad (k = 1, 2, \ldots, n-1),$$

are satisfied, which together with (19) gives

$$\|L\psi_j - \lambda\psi_j\| < \eta \qquad (j = 1, 2, \ldots).$$

From this inequality, seeing that $\eta > 0$ is arbitrary, we obtain the desired relation $\lambda \in C(\tilde{L})$.

Clearly, this result remains in force if in (19) and (20) we take $x \in J'$.

Observe that, except for the two-term operation, conditions (19) and (20) do not mean that a perturbation of the first term in (1) by the other terms is small in $\mathfrak{S}_2(\gamma, \infty)$ for sufficiently large γ. On the contrary, for any γ this perturbation is an unbounded operator. A more subtle criterion for the semi-axis $\lambda \geqslant 0$ to belong to the set $C(\tilde{L})$ will be given in No. 43.

The following two theorems are generalizations of Theorems 21 and 22 of the previous chapter.

Theorem 7 /31 (9)/. If $p_0(x) \equiv 1$ and if for any $\delta > 0$

$$\int_{M_{k,\delta}} |q_k(x)|^2\,dx < \infty \qquad (k = 2, 3, \ldots, 2n), \tag{21}$$

where $M_{k,\delta}$ is the set of values of x for which $|q_k(x)| > \delta$, then the positive part of the spectrum of the operator $\tilde{L}$ covers the entire semi-axis $\lambda \geqslant 0$.

Proof. Using again the test system of functions defined by formulas (72) and (73) of Chapter II, we obtain for $\lambda > 0$ and all natural numbers j,

starting with some $j=m$, that

$$\|L\varphi_j-\lambda\varphi_j\|\leqslant\|(-1)^n\varphi_j^{(2n)}-\lambda\varphi_j\|+\delta\sum_{k=2}^{2n}\|\varphi_j^{(2n-k)}\|+$$

$$+\sum_{k=2}^{2n}\sup_{(0<x<\infty)}|\varphi_j^{(2n-k)}(x)|\left\{\int\limits_{J_N\cap M_{k,\delta}}|q_k(x)|^2dx\right\}^{1/2}\leqslant$$

$$\leqslant\rho(\varepsilon,\delta)\|\varphi_j\|, \qquad (22)$$

where $J_N=(N,\infty)$. The functions $\varphi_j(x)$ for $j\geqslant m$ are N-finite and

$$\rho(\varepsilon,\delta)=\eta+A\delta+B\sqrt{\varepsilon}\max_{(k)}\left\{\int\limits_{J_N\cap M_{k,\delta}}|q_k(x)|^2dx\right\}^{1/2},$$

the numbers A and B being independent of ε and δ.

It follows from inequality (22) that

$$C(\tilde{L})\cap[\lambda-\rho(\varepsilon,\delta),\ \lambda+\rho(\varepsilon,\delta)]\neq\emptyset, \qquad (23)$$

but, by virtue of (21), we can choose the number N so that

$$\max_{(k)}\int\limits_{J_N\cap M_{k,\delta}}|q_k(x)|^2dx<1,$$

and consequently, $\rho(\varepsilon,\delta)$ together with ε and δ can be made arbitrarily small. From (23) we therefore have that $\lambda\in C(\tilde{L})$. Q. E. D.

Theorem 8. If for some sequence of intervals Δ_j whose length $|\Delta_j|$ increases without bound

$$\lim_{j\to\infty}\frac{1}{|\Delta_j|}\int\limits_{\Delta_j}|q_k(x)|^2dx=0 \qquad (k=2,\ 3,\ \ldots,\ 2n),$$

then the positive part of the spectrum of the operator $\tilde{L}$ covers the entire semi-axis $\lambda\geqslant 0$.

Proof. For the normalized test system of functions $\varphi_j(x)=\varphi(x-\alpha_j;\ \lambda,\ \eta_j)$ defined by formulas (72) and (73) of Chapter II we have

$$\|L\varphi_j-\lambda\varphi_j\|\leqslant\|(-1)^n\varphi_j^{(2n)}-\lambda\varphi_j\|+\sum_{k=2}^{2n}\left\|q_k\frac{d^{(2n-k)}\varphi_j}{dx^{2n-k}}\right\|\leqslant$$

$$\leqslant\eta_j\|\varphi_j\|+B\max_{(k)}\frac{1}{2|\Delta_j|}\int\limits_{\Delta_j}|q_k(x)|^2dx,$$

where the number B does not depend on j. For $j\to\infty$ we obtain from this inequality

$$\lim_{j\to\infty}\|L\varphi_j-\lambda\varphi_j\|=0$$

hence it follows that $\lambda\in C(\tilde{L})$.

43. Birman's theorem. This theorem contains Theorem 25 of No. 31 as a particular case and is based on an investigation of relatively complete continuity. It substantially improves the first of the criteria of No. 42. Its proof is based on the following compactness criterion, which is a generalization of Lemma 2 of No. 31.

Lemma 1 /13 (5)/. Let B be the minimum unclosed symmetric operator defined by the operation

$$(-1)^n \frac{d^{2n}}{dx^{2n}} y + y \tag{24}$$

on the semi-axis $x \geqslant 0$, and let K be a symmetric operator defined on $\mathfrak{D}_B$ by an operation of the form

$$(-1)^k \frac{d^k}{dx^k} \rho(x) \frac{d^k}{dx^k} y,$$

where $\rho(x) \geqslant 0$ and $k < n$.

In order that any B-bounded set be $K\sim$ compact, it is necessary and sufficient that for some (and consequently for any) $\omega > 0$ the limit relation

$$\lim_{x \to \infty} \int_x^{x+\omega} \rho(t)\, dt = 0 \tag{25}$$

is satisfied.

Proof for $k = 0$ is the same as that given in No. 31. In the general case we set

$$z(x) = y^{(k)}(x).$$

If some set $\mathfrak{M}$ of functions $y(x)$ is B-bounded, then, from inequality (6) of No. 26, the corresponding set of functions $z(x)$ is bounded in the metric of the functional

$$\int_0^\infty (|z^{(n-k)}|^2 + |z|^2)\, dx$$

and if condition (25) is satisfied, then from Lemma 2 of No. 31 it is compact in the metric of the functional

$$\int_0^\infty \rho(x) |z|^2\, dx,$$

that is, the set $\mathfrak{M}$ is $K\sim$ compact.

The proof of the necessity of limit relation (25) is carried out as in the case of Lemma 2 of No. 31, by subjecting the infinite system of finite functions with disjoint supports $\{y_j(x)\}$ to the conditions

$$y_j^{(k)}(x) \equiv 1 \qquad (x_j \leqslant x \leqslant x_j + \omega).$$

Theorem 9 /13 (5)/. If for some $\omega > 0$ the coefficients of differential operation (1) satisfy the conditions

$$0 < \alpha \leqslant p_0(x) \leqslant \beta < +\infty, \tag{26}$$

$$\lim_{x \to \infty} \int_x^{x+\omega} |p_0(t) - 1|\, dt = 0, \tag{27}$$

$$\lim_{x \to \infty} \int_x^{x+\omega} |p_k(t)|\, dt = 0 \qquad (k = 1, 2, \ldots, n), \tag{28}$$

then the corresponding operator $\tilde{L}$ is bounded from below and the continuous part of its spectrum coincides with the semi-axis $\lambda \geqslant 0$.

Proof for $p_0(x) \equiv 1$ follows directly from the previous lemma. In this case, according to Theorem 17 of No. 4, and from (28) the operators defined by each of the operations

$$(-1)^k \frac{d^k}{dx^k} p_{n-k}(x) \frac{d^k}{dx^k} y \qquad (k = 0, 1, \ldots, n-1), \tag{29}$$

are completely continuous relative to the operator B. Their sum is therefore also completely continuous relative to the operator B. But then from Lemma 1 of No. 4 the operator L is semi-bounded, and from Theorem 19 of No. 5 the continuous part of its spectrum coincides with the continuous part of the spectrum of the differentiation operator of $2n$-th order, that is, with the semi-axis $\lambda \geqslant 0$. Q. E. D.

To allow for the effect of the perturbation $p_0(x) - 1$ introduced into the leading coefficient of operation (24) defining the operator B, we make use of Theorem 20 of No. 5. The role of A is then played by the minimum unclosed symmetric operator defined on the semi-axis $x \geqslant 0$ by the operation

$$(-1)^n \frac{d^n}{dx^n} p_0(x) \frac{d^n}{dx^n} y + y.$$

Furthermore, the operator K is the difference

$$K = B - A,$$

The minimum unclosed symmetric operator defined on the semi-axis $x \geqslant 0$ by the operation

$$\frac{d^n}{dx^n} |1 - p_0(x)| \frac{d^n}{dx^n}$$

may then be taken as the majorizing operator K_1.

By virtue of Theorem 20 of No 5, it is sufficient to show the equivalence of the A- and B-metrics, the A-boundedness of the functional $(K_1 f, f)$, and the $\tilde{K}_1$-compactness of the transform $\mathring{B}^{-1}F$ of any set of functions F bounded in $\mathscr{L}_2(0, \infty)$.

The equivalence of the A- and B-metrics follows from double inequality (26). From this inequality it also follows that the quadratic functional $(K_1 f, f)$ is A-bounded.

Turning to the proof of the fundamental property of the operator K_1 we set, for $f \in F$,

$$\mathring{B}^{-1} f = y,$$

so that

$$(-1)^n y^{(2n)} + y = f(x). \tag{30}$$

The lower bound of the operator $\mathring{B}$ is obviously equal to unity, so that

$$\|y\|^2 \leqslant (\mathring{B}y, y) = (f, y) \leqslant \|f\| \cdot \|y\|,$$

that is,

$$\|y\| \leqslant \|f\|, \tag{31}$$

and then from (30) it follows that

$$\|y^{(2n)}\| \leqslant 2\|f\|. \tag{32}$$

Since the set F is bounded in $\mathscr{L}_2(0, \infty)$, we have for $f \in F$ from (31) and (32) that

$$\int_0^\infty (|y^{(2n)}|^2 + |y|^2)\, dx \leqslant C.$$

This inequality shows that functions $y(x)$ are bounded in the space $W_2^{(2n)}$. From condition (27) we then conclude, on the basis of Lemma 1 applied to the differential operators B of the order $4n$ and K of the order n, that the set $\mathring{B}^{-1}F$ is $\tilde{K}_1$-compact. But then, from Theorem 20 of No. 5, we have $C(B)=C(A)$, so that the continuous part of the spectrum of the positive operator defined by the operation

$$(-1)\frac{d^n}{dx^n}p_0(x)\frac{d^n}{dx^n}. \tag{33}$$

coincides with the semi-axis $\lambda \geqslant 0$.

The addition of terms of the form (29) to operation (33) leaves, when $k<n$, the corresponding operator bounded from below, and does not modify the continuous part of its spectrum. This is so because the operators defined by operation (29) are completely continuous relative to the operator B by virtue of condition (28). Consequently, from the equivalence of the A- and B-metrics, they are completely continuous relative to A. Q. E. D.

44. M. G. Krein's theorem on the multiplicative structure of positive differential operators. Let the coefficients $p_k(x)$ of the differential operation (1) belong to $C_{n-k}(0, \infty)$ and let

$$P[u, v]=\sum_{j, k=0}^{2n-1} a_{jk}(x)\, u^{(j)}v^{(k)} \tag{34}$$

be a bilinear form defined by Lagrange's identity

$$ul[v]-vl[u]=\frac{d}{dx}P[u, v]. \tag{35}$$

In (34), $a_{jk}=0$ for $j+k \geqslant 2n$ and $a_{n-1, n}=(-1)^n p_0(x)$.

From relation (35), for any pair of solutions $u(x)$ and $v(x)$ of the homogeneous equation

$$l[y]=0 \tag{36}$$

we have the identity

$$P[u, v] \equiv C \qquad (0 \leqslant x < \infty). \tag{37}$$

In particular, if $u(x)$ and $v(x)$ are two solutions of equation (36) satisfying the conditions

$$y(0)=y'(0)=\ldots=y^{(n-1)}(0)=0, \tag{38}$$

then it follows from (34) that for $x=0$ we have $P[u, v]=0$. But then from (37) we have, for all $x \geqslant 0$,

$$P[u, v] \equiv 0. \tag{39}$$

The two solutions of equation (36) satisfying condition (39) are called conjugate. There exist systems of n linearly independent pairwise conjugate solutions of equation (36). An example of such a system is any base of the n-dimensional manifold of the solutions of equation (36) satisfying conditions (38).

Let $\omega_1(x), \omega_2(x), \ldots, \omega_n(x)$ be an arbitrary system of linearly independent and pairwise conjugate solutions of equation (36). Denoting the Wronskian of the system of functions by w, we introduce a differential operation of n-th order

$$\mu[y]=\frac{w[\omega_1, \omega_2, \ldots, \omega_n; y]}{w[\omega_1, \omega_2, \ldots, \omega_n]} \tag{40}$$

and denote its conjugate in the sense of Lagrange by $\mu^*[y]$.

According to Frobenius' classical theorem (/98/, see also /3/), the operation $l[y]$ has the multiplicative representation

$$l[y]=\mu^* p_0 \mu[y], \tag{41}$$

which applies at any point $x \geqslant 0$ where

$$w[\omega_1, \omega_2, \ldots, \omega_n] \neq 0. \tag{42}$$

It follows from (40) that the coefficients of the differential operation

$$\mu[y] = y^{(n)} + q_1 y^{(n-1)} + \ldots + q_n y$$

will be n-fold continuously differentiable at the points where condition (42) is satisfied.

If (42) is satisfied for all $x > 0$, then the positiveness of the operator L follows directly from representation (41). M.G. Krein has established /53 (1)/ the converse proposition: if the operator L is positive, there exists a system of n linearly independent pairwise conjugate solutions of equation (36) satisfying condition (42) for all $x > 0$, so that multiplicative representation (41) also holds for all $x > 0$. We give below the proof of this theorem, and also M.G. Krein's proof of Frobenius' theorem. The transition from the finite interval considered in /53 (1)/ to the semi-axis $x \geqslant 0$ requires almost no modifications in the arguments of /53 (1)/.

Theorem 10 /98/. For every system of n linearly independent solutions $\omega_1(x), \omega_2(x), \ldots, \omega_n(x)$ of equation (36) satisfying for $x \geqslant 0$ the condition

$$P[\omega_j, \omega_k] = 0 \qquad (j, k = 1, 2, \ldots, n), \tag{43}$$

there exists multiplicative representation (41), valid at any point $x \geqslant 0$ where inequality (42) is satisfied.

Proof /53 (1)/. Let $V(x, s)$ be the Cauchy function of the operation $\mu[y]$, so that

$$V(x, s) = \sum_{k=1}^{n} \omega_j^*(s)\, \omega_j(x) \tag{44}$$

and

$$\mu^*[\omega_j^*] = 0 \qquad (j = 1, 2, \ldots, n). \tag{45}$$

Furthermore, suppose that condition (42) is satisfied for $x_0 < x < a$ (it is shown in /3/ that the zeros of the Wronskian w do not have finite limit points).

Since the transposed kernel $V(s, x)$ is the Cauchy function of the operation $\mu^*[y]$, then for any function $f(x) \in C(x_0, a)$ we have

$$\mu^*\left\{\int_{x_0}^{x} V(s, x) f(s)\, ds\right\} = f(x)$$

or, for $f(x) = l[y]$,

$$l[y] = \mu^*\left\{\int_{x_0}^{x} V(s, x)\, l_s[y]\, ds\right\}. \tag{46}$$

We now show that representation (46) coincides with the required representation (41).

Setting $z(s) = V(s, x)$ in the identity

$$\int_{x_0}^{x} z(s)\, l_s[y]\, ds = P[y, z]\Big|_{x_0}^{x} + \int_{x_0}^{x} l_s[z]\, y(s)\, ds$$

and seeing that then $l_s[z] = 0$, we obtain from (44) that

$$\int_{x_0}^{x} V(s, x)\, l_s[y]\, ds = P_s[V(s, x), y(s)]\Big|_{s=x_0}^{s=x} =$$

$$= \left\{\sum_{j=1}^{n} P_s[\omega_j, y]\, \omega_j^*(x)\right\}\Big|_{s=x_0}^{s=x} = m[y] - \sum_{j=1}^{n} c_j \omega_j^*(x), \tag{47}$$

where

$$m[y] = \sum_{j=1}^{n} P_x[\omega_j, y]\, \omega_j^*(x). \tag{48}$$

Substituting (47) in (46) and applying equalities (45), we obtain the representation of the operation $l[y]$ in the form

$$l[y] = \mu^* m[y].$$

It remains to show that the differential operations of n-th order $m[y]$ and $(-1)^n p_0 \mu[y]$ are identical.

Since by virtue of (43) and (48) the fundamental system $\omega_1(x), \omega_2(x), \ldots, \omega_n(x)$ of the equation $\mu[y] = 0$ is also the fundamental system of the equation $m[y] = 0$, it is sufficient to show that the leading coefficients of the operations $m[y]$ and $(-1)^n p_0 \mu[y]$ are identical. Now, by virtue of (34) and (48) the coefficient of $y^{(n)}$ in (48) is

$$a_{n-1,n}(x)\left\{ V(s, x)\Big|_{s=x} + \frac{\partial V(s, x)}{\partial s}\Big|_{s=x} + \ldots \right.$$
$$\left. \ldots + \frac{\partial^{n-2} V(s, x)}{\partial s^{n-2}}\Big|_{s=x} + \frac{\partial^{n-1} V(s, x)}{\partial s^{n-1}}\Big|_{s=x} \right\},$$

and since

$$\frac{\partial^k V(s, x)}{\partial s^k}\Big|_{s=x} = \begin{cases} 0 & (0 \leqslant k \leqslant n-2), \\ 1 & (k = n-1) \end{cases}$$

and

$$a_{n-1,n}(x) = (-1)^n p_0(x),$$

the leading coefficients of the operations $m[y]$ and $(-1)^n p_0 \mu[y]$ are identical.

Frobenius' theorem is thus proved.

Theorem 11 /53 (1)/. If the operator L is positive in $\mathfrak{L}_2(0, \infty)$, there exists a system of linearly independent solutions of equation (36) $\omega_1(x), \omega_2(x), \ldots, \omega_n(x)$, satisfying condition (43) and the following condition is satisfied for all $x > 0$:

$$w[\omega_1, \omega_2, \ldots, \omega_n] \neq 0.$$

Proof. Let a regular self-adjoint operator L_a be defined in $\mathfrak{L}_2(0, a)$ for some $a > 0$ by means of the boundary conditions

$$y^{(k)}(0) = y^{(k)}(a) = 0 \qquad (k = 0, 1, \ldots, n-1)$$

Since the operator L is positive, then the operator L_a is positive definite. Its Green's function has the form

$$K(x, s) = \begin{cases} \sum_{j=1}^{n} \psi_j(x) \chi_j(s) & (x \leqslant s), \\ \sum_{j=1}^{n} \psi_j(s) \chi_j(x) & (x \geqslant s), \end{cases} \tag{49}$$

where $\psi_1(x), \psi_2(x), \ldots, \psi_n(x)$ is some system of linearly independent solutions of equation (36) which satisfy the initial conditions

$$y(0) = y'(0) = \ldots = y^{(n-1)}(0) = 0. \tag{50}$$

On the other hand, according to Merser's theorem, we have, in the sense of uniform convergence, that

$$K(x, s) = \sum_{k=1}^{\infty} \frac{\varphi_k(x) \varphi_k(s)}{\lambda_k}, \tag{51}$$

where $\{\varphi_k(x)\}_{k=1}^{\infty}$ is the complete orthonormalized system of eigenfunctions of the positive kernel $K(x, s)$, and $\{\lambda_k\}_{k=1}^{\infty}$ is the corresponding sequence of eigenvalues.

Differentiating the two sides of (51) p times with respect to x and q times with respect to s ($p, q = 0, 1, \ldots, n-1$), we obtain (see /37/, /38/) the expansions

$$K_{pq}(x, s) = \sum_{k=1}^{\infty} \frac{\varphi_k^{(p)}(x) \varphi_k^{(q)}(s)}{\lambda_k}, \tag{52}$$

which converge absolutely and uniformly in the square $0 \leqslant x, s \leqslant a$.

We introduce the quadratic form of n variables

$$\sum_{p,\,q=0}^{n-1} K_{pq}(x,\,x)\,\xi_p\xi_q \qquad (0 < x < a) \tag{53}$$

The positiveness of this quadratic form is obvious, since from (52) it follows that

$$\sum_{p,\,q=0}^{n-1} K_{pq}(x,\,x)\,\xi_p\xi_q = \sum_{k=1}^{\infty} \frac{\left\{\sum_{p=0}^{n-1} \varphi_k^{(p)}(x)\,\xi_p\right\}^2}{\lambda_k} \tag{54}$$

Its positiveness can easily be derived in the limit directly from the positiveness of the kernel $K(x,\,s)$, without expansion (51). From (54), however, it easily follows that form (53) is not only positive, but also positive definite.

In fact, from the equality

$$\sum_{p,\,q}^{n-1} K_{pq}(s_0,\,s_0)\,\overset{0}{\xi}_p\overset{0}{\xi}_q \qquad (0 < s_0 < a)$$

and (54) it follows that for any k

$$\sum_{p=0}^{n-1} \overset{0}{\xi}_p \varphi_k^{(p)}(s_0) = 0.$$

Hence, again using expansion (52) for $q = 0$, we obtain

$$\sum_{p=0}^{n-1} \overset{0}{\xi}_p K_{0p}(x,\,s_0) = \sum_{k=1}^{\infty} \frac{\varphi_k(x)\left\{\sum_{p=0}^{n-1} \overset{0}{\xi}_p \varphi_k^{(p)}(s_0)\right\}}{\lambda_k} = 0, \tag{55}$$

which holds identically in x in the interval $0 < x < a$.

On the other hand, using the well-known properties of Green's functions, it is easy to obtain the relations

$$K_{rp}(s+0,\,s) - K_{rp}(s-0,\,s) = \begin{cases} 0 & (r+p < 2n-1), \\ \dfrac{(-1)^p}{p_0(s)} & (r+p = 2n-1). \end{cases} \tag{56}$$

Setting

$$g(x) = \sum_{p=0}^{n-1} \overset{0}{\xi}_p K_{0p}(x,\,s_0), \tag{57}$$

we have, from (55), that

$$g(x) \equiv 0. \tag{58}$$

However, differentiating the two sides of (57) n times with respect to x and using relations (56), we obtain

$$g^{(n)}(s_0+0) - g^{(n)}(s_0-0) = \frac{\overset{0}{\xi}_{n-1}}{p_0(s_0)},$$

so that $\overset{0}{\xi}_{n-1} = 0$. Now, differentiating the two sides of (57) $n+1$ times with respect to x and using relations (56) and (58), we also obtain that $\overset{0}{\xi}_{n-2} = 0$.

Iterating, we finally obtain

$$\overset{0}{\xi}_0 = \overset{0}{\xi}_1 = \ldots = \overset{0}{\xi}_{n-1} = 0,$$

thus proving that quadratic form (53) is positive definite.

Since the form (53) is positive definite, we have

$$\operatorname{Det} \| K_{pq}(x, x) \|_{p, q=0}^{n-1} > 0 \qquad (0 < x < a). \tag{59}$$

But for p and q smaller than n we have from (49) that

$$K_{pq}(x, x) = \sum_{j=1}^{n} \psi_j^{(p)}(x) \chi_j^{(q)}(x),$$

so that we have the matrix equality

$$\| K_{pq}(x, x) \|_{p, q=0}^{n-1} = \| \psi_j^{(p)} \| \| \chi_j^{(q)} \|^*. \tag{60}$$

It follows from relations (60) and (59) that for $0 < x < a$

$$w[\psi_1, \psi_2, \ldots, \psi_n] w[\chi_1, \chi_2, \ldots, \chi_n] = \operatorname{Det} \| K_{pq}(x, x) \|_{p, q=0}^{n-1} > 0,$$

leading to the relation

$$w[\psi_1, \psi_2, \ldots, \psi_n] \neq 0 \qquad (0 < x < a). \tag{61}$$

Let $\omega_1(x), \omega_2(x), \ldots, \omega_n(x)$ be any basis of the n-dimensional manifold of the solutions of equation (36) satisfying initial equations (38). As was noted at the beginning of this section, the functions $\omega_1(x), \omega_2(x), \ldots, \omega_n(x)$, are pairwise conjugate, that is,

$$P[\omega_j, \omega_k] \equiv 0 \qquad (0 \leqslant x < \infty;\ j, k = 1, 2, \ldots, n).$$

We now show that for any $x_0 > 0$ we have

$$w[\omega_1, \omega_2, \ldots, \omega_n] \neq 0. \tag{62}$$

For this purpose we choose for a given x_0 a number $a > x_0$ and construct the Green's function of the operator L_a according to formulas (49). Relation (61) is then satisfied for the functions $\psi_1(x), \psi_2(x), \ldots, \psi_n(x)$ which depend on a. Now, the functions $\omega_1(x), \omega_2(x), \ldots, \omega_n(x)$ are linear combinations of the functions $\psi_1(x), \psi_2(x), \ldots, \psi_n(x)$, so that

$$w[\omega_1, \omega_2, \ldots, \omega_n] = \gamma w[\psi_1, \psi_2, \ldots, \psi_n] \qquad (0 < x < a),$$

where the constant γ is not zero since the functions $\omega_1(x), \omega_2(x), \ldots, \omega_n(x)$ are linearly independent.

Inequality (61) is thus satisfied for all positive values of x smaller than a, and in particular for $x = x_0$. Q. E. D.

The following theorem on the multiplicative structure of positive differential operators mentioned in No. 40 follows from theorems 10 and 11.

Theorem 12 /53 (1)/. A necessary and sufficient condition for the operator L to be positive in $\mathscr{D}_2(0, \infty)$ is that the operation l may be represented in the form

$$l[y] = \mu^* p_0 \mu[y] \qquad (0 < x < \infty),$$

where

$$\mu[y] = y^{(n)} + q_1(x) y^{(n-1)} + \ldots + q_n(x) y$$

is an operation with the coefficients $q_k(x) \in C_n(0, \infty)$ $(k = 1, 2, \ldots, n)$ and $\mu^*[y]$ is its Lagrange conjugate operation.

To conclude we observe that a detailed investigation of the systems of conjugate solutions of the corresponding Wronskians $w[\omega_1, \omega_2, \ldots, \omega_n]$ was carried out in a recently published paper by E. Heinz /99/. Van-der-Waerden, in his introduction to this paper, notes that it constitutes the formulation and completion of Rellich's investigations /83 (5)/. In particular, it is interesting to note the following theorem of Heinz-Rellich, which is a generalization of Sturm's classical theorem on the alternation of the zeros of the solutions of the second-order differential equation.

Theorem 13 /99/. If the Wronskian of some system of n linearly independent and pairwise conjugate solutions of equation (36) has n zeros in some interval $[\alpha, \beta]$, then the Wronskian of any other system of n pairwise conjugate solutions of this equation has at least one zero in this interval.

From this theorem the second part of the following proposition on the relation between the zeros of the Wronskians $w[\omega_1, \omega_2, \ldots, \omega_n]$ and the negative part of the spectrum of the operators $\tilde{L}$ may be proved.

Theorem 14. If the Wronskian of any system of n linearly independent pairwise conjugate solutions of equation (36) has a finite number of zeros on the semi-axis $x > 0$, then the negative part of the spectrum $S(\tilde{L})$ is a finite set.

Conversely, if the negative part of the spectrum $S(\tilde{L})$ is a finite set, then the Wronskian of any system of n linearly independent pairwise conjugate solutions of equation (36) has only a finite number of zeros.

Proof. If the number of zeros of the Wronskian $w[\omega_1, \omega_2, \ldots, \omega_n]$ is finite, there exists a number $N \geqslant 0$ such that for $x > N$ we have

$$w[\omega_1, \omega_2, \ldots, \omega_n] \neq 0.$$

Therefore, from Frobenius' theorem, operation (l) may be represented in the form (41) on the semi-axis $x > N$, and the operator generated by operation (l) on the semi-axis $x > N$ is thus positive. Hence, from Theorem 28 of No. 10 it follows that the set of points of the negative part of the spectrum $S(\tilde{L})$ is finite.

Conversely, if the negative part of the spectrum $S(\tilde{L})$ consists of a finite number of points, then from Theorem 28 of No. 10 it follows that the operator L_N is positive for some $N \geqslant 0$. But then, according to M. G. Krein's theorem, the Wronskian of some system of n linearly independent pairwise conjugate solutions of equation (36) does not vanish for $x > N$. Consequently, it will have only a finite number of zeros on the semi-axis $x > 0$. Now, from Theorem 13 of Heinz-Rellich it follows that the Wronskian of any system of n linearly independent pairwise conjugate solutions of equation (36) has only a finite number of zeros on the semi-axis $x > 0$. This completes the proof of the theorem.

Chapter IV

THE SPECTRUM OF MULTIDIMENSIONAL DIFFERENTIAL OPERATORS

45. **Multidimensional differential operations.** This chapter gives an account of the methods of investigating the character of the spectrum of self-adjoint operators $\tilde{L}$ generated in $\mathfrak{L}_2(\Omega)$ by the differential operation

$$l[u] = -\Delta u + q(P)u, \tag{1}$$

where the potential $q(P)$ is assumed continuously differentiable. The domain Ω may coincide with the whole Euclidean space $\mathfrak{E}_m$ or be a proper part of it. In the latter case the potential is assumed continuously differentiable up to the boundary S of the domain Ω, so that the singularity of the operation (1) is due only to the unboundedness of the domain Ω.

The theorems of Nos. 46—50 may be carried over in an appropriate formulation to the more general operation

$$(-1)^n \Delta^n u + q(P)u. \tag{2}$$

Apart from the propositions on the character of the spectrum as related to the behavior of the potential, which lead to generalizations of the one-dimensional criteria of Chapter II, we also consider here the influence of the form of the domain on the spectrum, a question which is characteristic of the multidimensional case.

As before (see the footnote on p. 47), in order to avoid specifying explicitly the class of domains or surfaces which are considered below, we use the term "any", although any domain or surface considered is subject to some rather general conditions. Moreover, the statement that there does not exist a solution of a homogeneous equation, belonging to a given class, means the absence of a non-trivial solution in this class.

Some further problems of qualitative spectral analysis of elliptic boundary-value problems are considered in a recent paper /13 (7)/ by M. Sh. Birman, where they are systematically related to the corresponding inclusion theorems. On the spectrum of non-elliptic problems see reference /75/. For other papers on qualitative spectral analysis of multidimensional boundary-value problems we mention /16/, /17/, and /25/.

46. **Theorems on the discrete part of the spectrum.** In the present subsection we shall consider semi-bounded operators. As in the one-dimensional case, the simplest criterion here is the following:

Theorem 1 /97 (1)/. If

$$\lim_{|OP|\to\infty} \inf q(P) = \mu > -\infty,$$

then $L = L^*$ and on the semi-axis $(-\infty, \mu)$ the spectrum of the operator L contains only eigenvalues of finite multiplicity with the only possible limit point $\lambda = \mu$.

Proof /31 (5)/. The first part of the theorem follows from Theorem 34 of No. 16. Proceeding to the proof of the second part, we assume for definiteness that $\mu < \infty$, and choose the radius $R = R_\varepsilon$ of the decomposition sphere $|OP| = R_\varepsilon$ such that for $|OP| > R_\varepsilon$ we have $q(P) > \mu - \varepsilon$, where $\varepsilon > 0$.

According to Theorem 36 of No. 17 we have

$$C(L) = C(L_R),$$

where L_R is the closure of the operator $\dot{L}_R$, defined by the operation (1) for $|OP| > R_\varepsilon$ on twice continuously differentiable functions which vanish in the neighborhood of the decomposition surface $|OP| = R_\varepsilon$.

For $\varphi \in \mathfrak{D}_{L_R}$ we have

$$(L_R\varphi, \varphi) > (\mu - \varepsilon)(\varphi, \varphi),$$

so that

$$C(L_R) \cap (-\infty, \mu - \varepsilon) = \emptyset$$

and, consequently,

$$C(L) \cap (-\infty, \mu - \varepsilon) = \emptyset,$$

hence, since the number $\varepsilon > 0$ is arbitrary, we obtain the relation

$$C(L) \cap (-\infty, \mu) = \emptyset.$$

Since the boundedness of the operator L from below implies that its spectrum $S(L)$ is bounded from below, the theorem is proved.

An integral criterion for the boundedness and discreteness of the negative part of the spectrum is given by the following theorem.

Theorem 2 /31 (10)/. If for any $\delta > 0$ the integral

$$\int_0^\infty |q_\delta^*(P)|\, dr, \tag{3}$$

converges, where

$$q_\delta^*(P) = \begin{cases} q^*(P) & (|q^*(P)| > \delta), \\ 0 & (|q^*(P)| \leqslant \delta) \end{cases}$$

and

$$q^*(P) = \min\{q(P), 0\},$$

then $L = L^*$ and the negative part of the spectrum of the operator L is bounded and discrete.

Proof. We transform the quadratic functional

$$\Phi_\varepsilon[u] = \int_\Omega |\nabla u|^2\, d\omega + \int_\Omega q\,|u|^2\, d\omega + \varepsilon \int_\Omega |u|^2\, d\omega$$

corresponding to operation (1) by means of the substitution

$$u = r^{-1}v, \tag{4}$$

where u is an arbitrary real function from $\mathfrak{D}_L$. We have

$$|\nabla u|^2 = \frac{1}{r^2}|\nabla v|^2 + \frac{1}{r^4}v^2 - \frac{2v}{r^3}\frac{\partial v}{\partial r},$$

and integration by parts gives

$$\int_{\mathfrak{S}} \frac{1}{r^3} 2v \frac{\partial v}{\partial r} d\omega = \int_{-\pi/2}^{\pi/2} \sin\vartheta\, d\vartheta \int_0^{2\pi} d\varphi \int_0^{\infty} \frac{1}{r}\frac{\partial v^2}{\partial r} dr =$$

$$= \int_{-\pi/2}^{\pi/2}\int_0^{2\pi}\int_0^{\infty} \frac{v^2}{r^2}\sin\vartheta\, dr\, d\varphi\, d\vartheta,$$

so that

$$\Phi_\varepsilon[u] = \int_{\mathfrak{S}} \{|\nabla v|^2 + (q+\varepsilon)v^2\}\frac{d\omega}{r^2}.$$

For any ε $(0<\varepsilon<1)$ we choose, using the convergence of the integral (3), a number $N = N_\varepsilon$ such that for the domain $0 \leqslant \varphi < 2\pi, -\frac{\pi}{2} \leqslant \vartheta \leqslant \frac{\pi}{2}$ we have

$$\int_N^{\infty} |q^*(r, \varphi, \vartheta)|\, dr < \frac{\varepsilon}{4}. \tag{5}$$

We shall show that the functional

$$\Phi_\varepsilon^*[u] = \int_{-\pi/2}^{\pi/2}\int_0^{2\pi} \sin\vartheta\, d\varphi\, d\vartheta \int_N^{\infty} \{|\nabla v|^2 + (q^*+\varepsilon)v^2\}\, dr \tag{6}$$

is non-negative on any N-finite function $u \in \mathfrak{D}_L$. Let

$$\hat{v}(\varphi, \vartheta) = \max_{N<r<\infty} |v(r, \varphi, \vartheta)|$$

and let $R(\varphi, \vartheta)$ be the distance from the origin of the coordinates to the nearest maximum point of the modulus of the N-finite function $v(r, \varphi, \vartheta)$ for $N < r < \infty$. Without loss of generality, it is possible to regard the function $v(r, \varphi, \vartheta)$ as normalized, so that

$$\int_{-\pi/2}^{\pi/2}\int_0^{2\pi}\int_N^{\infty} |v|^2 \sin\vartheta\, dr\, d\varphi\, d\vartheta = 1. \tag{7}$$

Now, applying the Cauchy-Bunyakovskii inequality twice, we obtain

$$\frac{1}{2}\int_{-\pi/2}^{\pi/2}\int_0^{2\pi} \hat{v}^2(\varphi, \vartheta)\sin\vartheta\, d\varphi\, d\vartheta = -\int_{-\pi/2}^{\pi/2}\int_0^{2\pi} \sin\vartheta\, d\varphi\, d\vartheta \int_{R(\varphi, v)}^{\infty} v\frac{\partial v}{\partial r} dr \leqslant$$

$$\leqslant \int_{-\pi/2}^{\pi/2}\int_0^{2\pi}\left\{\int_{R(\varphi,\vartheta)}^{\infty} v^2 dr\right\}^{1/2}\left\{\int_{R(\varphi,\vartheta)}^{\infty}\left|\frac{\partial v}{\partial r}\right|^2 dr\right\}^{1/2}\sin\vartheta\, d\varphi\, d\vartheta \leqslant$$

$$\leqslant \left\{\int_{-\pi/2}^{\pi/2}\int_0^{2\pi}\int_{R(\varphi,\vartheta)}^{\infty} v^2 \sin\vartheta\, dr\, d\varphi\, d\vartheta\right\}^{1/2} \times$$

$$\times \left\{\int_{-\pi/2}^{\pi/2}\int_0^{2\pi}\int_{R(\varphi,\vartheta)}^{\infty}\left|\frac{\partial v}{\partial r}\right|^2 \sin\vartheta\, dr\, d\varphi\, d\vartheta\right\}^{1/2} \leqslant \left\{\int_{\mathfrak{S}}\left|\frac{\partial v}{\partial r}\right|^2 \frac{d\omega}{r^2}\right\}^{1/2}.$$

On the other hand, we obviously have

$$\int\limits_{(S)} |\nabla v|^2 \frac{d\omega}{r^3} \geqslant \int\limits_{(S)} \left|\frac{\partial v}{\partial r}\right|^2 \frac{d\omega}{r^3}.$$

We thus have the inequality

$$\int\limits_{-\pi/2}^{\pi/2}\int\limits_{0}^{2\pi}\int\limits_{N}^{\infty} |\nabla v|^2 \sin\vartheta\, dr\, d\varphi\, d\vartheta \geqslant$$

$$\geqslant \frac{1}{4}\left\{\int\limits_{-\pi/2}^{\pi/2}\int\limits_{0}^{2\pi} \hat{v}^2(\varphi, \vartheta)\sin\vartheta\, d\varphi\, d\vartheta\right\}^2, \tag{8}$$

similar to inequality (7) of No. 26.

Let us consider the following two cases separately:

$$1^\circ)\ \int\limits_{-\pi/2}^{\pi/2}\int\limits_{0}^{2\pi} \hat{v}^2(\varphi, \vartheta)\sin\vartheta\, d\varphi\, d\vartheta \leqslant 1,$$

$$2^\circ)\ \int\limits_{-\pi/2}^{\pi/2}\int\limits_{0}^{2\pi} \hat{v}^2(\varphi, \vartheta)\sin\vartheta\, d\varphi\, d\vartheta > 1.$$

In the first case, omitting the term $|\nabla u|^2$ in the integrand in (6) and using relations (5) and (7), we obtain

$$\Phi_\varepsilon[u] > \int\limits_{-\pi/2}^{\pi/2}\int\limits_{0}^{2\pi} \sin\vartheta\, d\varphi\, d\vartheta \int\limits_{N}^{\infty} qv^2 dr +$$

$$+\varepsilon \int\limits_{-\pi/2}^{\pi/2}\int\limits_{0}^{2\pi} \sin\varphi\, d\varphi\, d\vartheta \int\limits_{N}^{\infty} |v|^2 dr \geqslant$$

$$\geqslant -\int\limits_{-\pi/2}^{\pi/2}\int\limits_{0}^{2\pi} \hat{v}^2(r, \varphi)\sin\vartheta\, d\varphi\, d\vartheta \int\limits_{N}^{\infty} |q^*|\, dr + \varepsilon > 0.$$

In the second case, omitting the last term in the integrand in (6) and using inequalities (5) and (8), we obtain

$$\Phi_\varepsilon[u] > \frac{1}{4}\left\{\int\limits_{-\pi/2}^{\pi/2}\int\limits_{0}^{2\pi} \hat{v}^2(\varphi, \vartheta)\sin\vartheta\, d\varphi\, d\vartheta\right\}^{1/2} -$$

$$-\int\limits_{-\pi/2}^{\pi/2}\int\limits_{0}^{2\pi} \hat{v}^2(\varphi, \vartheta)\sin\vartheta\, d\varphi\, d\vartheta \int\limits_{N}^{\infty} |q^*|\, dr >$$

$$> \left\{\frac{1}{4}\int\limits_{-\pi/2}^{\pi/2}\int\limits_{0}^{2\pi} \hat{v}^2(\varphi, \vartheta)\sin\vartheta\, d\varphi\, d\vartheta -\right.$$

$$\left. -\frac{\varepsilon}{4}\right\} \int\limits_{-\pi/2}^{\pi/2}\int\limits_{0}^{2\pi} \hat{v}^2(\varphi, \vartheta)\sin\vartheta\, d\varphi\, d\vartheta > 0.$$

Thus, in both cases we have established the inequality

$$\Phi_\varepsilon[u] > 0, \tag{9}$$

from which, by virtue of the results of No. 19, it follows that the operator L

is bounded from below; consequently, it is also self-adjoint. From inequality (9) it also follows by Theorem 41 of No. 19 that the negative part of the spectrum of the operator L is discrete. The theorem is thus proved.

Theorem 2 holds, in particular, if for some $\nu \geqslant 1$ the integral

$$\int_0^\infty |q(P)|^\nu dr$$

converges for any values of the angular coordinates.

For $\nu = m = 1$ the latter criterion reduces to H. Weyl's well-known criterion cited in No. 27.

To prove Theorem 2 for any m, instead of (4) we should make the substitution

$$u = r^{\frac{1-m}{2}} v. \tag{10}$$

If the operator L is self-adjoint, semi-bounded, and has only discrete eigenvalues

$$\lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \leqslant \ldots, \tag{11}$$

to the left of the point $\lambda = \mu$, then from the spectral formula it follows immediately that R. Courant's variational principles hold for the eigenvalues $\{\lambda_k\}$.

In particular, we have

$$\lambda_k = \inf (Lu, u), \tag{12}$$

the infimum being over $u \in \mathfrak{D}_L$, $\|u\| = 1$, $(u, \varphi_j) = 0$ $(j = 1, 2, \ldots, k-1)$, where $\{\varphi_j\}$ is a sequence of eigenfunctions corresponding to the eigenvalues (11). According to the definition of the minimal operator L, the function u in (12) may be considered finite.

The following propositions are generalizations of R. Courant's well-known comparison theorems to singular boundary-value problems.

Lemma 1 /40/. Let $\{\lambda_j\}$ and $\{\lambda'_j\}$ be the spectra of the regular boundary-value problems, defined by operation (1) in a bounded domain ω with the conditions $u = 0$ and respectively $\frac{\partial u}{\partial n} = 0$ on the boundary S of this domain. Furthermore, let $\alpha > -\infty$ be the first point of the spectrum of the self-adjoint operator L_S in $\mathfrak{L}_2[S, \infty]$, defined by operation (1) and the boundary condition $\frac{\partial u}{\partial n}\Big|_{P \in S} = 0$. Then the number k of eigenvalues of the self-adjoint operator L in $\mathfrak{L}_2(\mathfrak{G}_m)$ which are smaller than α satisfies the inequality

$$n \leqslant k \leqslant n', \tag{13}$$

where the numbers n and n' are determined by the inequalities

$$\lambda_n < \alpha \leqslant \lambda_{n+1}, \quad \lambda'_{n'} < \alpha \leqslant \lambda'_{n'+1}.$$

Proof. Without loss of generality, we may take $m = 3$ and $\alpha = 0$. The eigenfunctions of the two regular boundary-value problems, corresponding to the negative eigenvalues $\{\lambda_j\}_{j=1}^{n}$ and $\{\lambda'_j\}_{j=1}^{n'}$, will be denoted by $\{\varphi_j(P)\}_{j=1}^{n}$ and $\{\psi_j(P)\}_{j=1}^{n'}$ respectively.

We first establish the left-hand side of inequality (13). Suppose that $k < n$; then there exists some non-trivial linear combination $u(P) = \sum_{j=1}^{n} C_j \varphi_j(P)$ orthogonal to all the eigenfunctions corresponding to the first k eigenvalues of the operator L. The function u satisfies

$$(L[u], u) < 0. \tag{14}$$

Smoothing the function u in accordance with Lemma 13 of No. 19, we obtain a function $\tilde{u} \in \mathfrak{D}_L$ which is orthogonal to all the eigenfunction corresponding to the negative eigenvalues of the operator L.

This function must satisfy

$$(L\tilde{u}, \tilde{u}) \geqslant 0,$$

whereas from (13) we obtain

$$(L\tilde{u}, \tilde{u}) < 0.$$

This contradiction proves the left-hand side of the double inequality (13).

Turning to the proof of the right-hand side of inequality (13), we denote the linearly independent eigenfunctions corresponding to its first k negative eigenvalues of the operator L by $v_1(P), v_2(P), \ldots, v_k(P)$.

Suppose that $k > n'$; then there exists some non-trivial linear combination $v(P) = \sum_{j=1}^{k} c_j v_j(P)$ satisfying the conditions

$$\int_{\omega} v(P)\,\overline{\psi_j(P)}\,d\omega = 0 \qquad (j = 1, 2, \ldots, n').$$

The function v then satisfies

$$(Lv, v) < 0,$$

that is

$$(L[v], v) = (l[v], v)_{\omega} + (l[v], v)_{\xi - \omega} < 0. \tag{15}$$

The second term $(l[v], v)_{\xi - \omega}$ cannot be negative, since then the quantity

$$\inf (l[v], v)_{\xi - \omega}, \quad v \in \mathfrak{D}_L, \quad \|v\| = 1, \tag{16}$$

would be negative. But the quantity (16) does not change if the function v is subjected to the natural boundary condition $\frac{\partial v}{\partial n}\Big|_S = 0$, so that the quantity

$$\inf (L_S v, v), \quad v \in \mathfrak{D}_{L_S}, \quad \|v\| = 1,$$

is negative, which implies the existence of a negative point in the spectrum of the operator L_S, contrary to the assumption.

Therefore, from (15) it follows that

$$(l[v], v)_{\omega} < 0.$$

Thus the minimum of the quadratic functional corresponding to the boundary-value problem in the domain ω with the conditions $(v, \psi_j) = 0$ $(j = 1, 2, \ldots, n')$ and the boundary condition $\frac{\partial u}{\partial n}\Big|_{P \in S} = 0$ would be negative, since this condition

is natural. But this minimum is equal to the number $\lambda'_{n'+1}$, which is non-negative.

This contradiction proves the right-hand side of the double inequality (13).

It is easy to generalize Lemma 1 in an appropriate form to the case when the number $\alpha > -\infty$ is the first point of the continuous part of the spectrum of the operator L_S, and is preceded only by a finite number of eigenvalues of finite multiplicity. By the results of No. 16, we may consider instead of the entire space $\mathfrak{E}$ a part of $\mathfrak{E}$ bounded by an infinite surface with the boundary condition $u = 0$ on this boundary.

Lemma 2. Let the minimal operators L_1 and L_2, defined by operation (1) in the domains Ω_1 and Ω_2 with zero boundary conditions on the boundary, be self-adjoint.

If the domain Ω_2 contains the domain Ω_1, then the smallest eigenvalue and the successive eigenvalues of the operator L_2 preceding the exact lower bound of the continuous part of its spectrum, and also this lower bound, cannot be greater than the corresponding quantities for the operator L_1.

Proof follows immediately from the minimax principle, from Lemma 13 of No. 19 and from the results of No. 3. The problem of the strict decrease in the smallest eigenvalue when the domain is extended, which was solved positively for bounded domains /64 (1)/, remains open under the conditions of Lemma 2, when the domains may be unbounded.

47. **On criteria for the semi-boundedness and discreteness of the spectrum of multidimensional boundary-value problems.** The simplest criterion of this kind is given by the following theorem, which is an immediate consequence of Theorem 1 of the previous subsection.

Theorem 3 /97 (1)/. If

$$\lim_{|OP|\to\infty} q(P) = \infty,$$

then the spectrum of the operator $L = L^*$ is bounded from below and discrete.

The proof of a more subtle criterion, corresponding to the one-dimensional Theorem 6 of No. 28, is associated with considerable difficulties due to the absence, in the case of functions of many variables, of an analogue of the inequality

$$\int_{\Omega_k} |y^2(x) - y^2(x_k)|\, dx \leqslant \omega \int_{\Omega_k} (y'^2 + y^2)\, dx, \tag{17}$$

used in No. 28. In his paper /70/, A. M. Molchanov succeeded in obtaining together with a discreteness criterion for the spectrum of the one-dimensional Schroedinger operator, a multidimensional generalization for the operator (1) in $\mathscr{L}_2(\mathfrak{E}_m)$ for any $m > 1$.

To formulate A. M. Molchanov's result, we recall the definition of M. V. Keldysh /51 (1)/ of the capacity of a set.

Let F be a closed bounded set in $\mathfrak{E}_m$ and μ—a completely additive positive measure, giving the distribution of charge on F. Let $u(P)$ denote the field potential of this distribution, so that

$$u(P) = \int_F \frac{d\mu(Q)}{|PQ|^{m-2}}.$$

The capacity $C(F)$ of a set F is defined as the maximum total charge for which $u(P) \leqslant 1$, that is

$$C(F) = \sup_{u \leqslant 1} \int_F d\mu(Q).$$

Moreover, the set F, situated in a cube $R \subset \mathfrak{E}_m$ with the edge ω, is said to be the non-essential cut of this cube, if its capacity satisfies the inequality

$$C(F) \leqslant \frac{|\omega|^{m-2}}{(4m)^{4m}}.$$

A. M. Molchanov's result states that when the potential $q(P)$ is bounded from below a necessary and sufficient condition for the discreteness of the spectrum of the Schroedinger operator in $\mathscr{L}_2(\mathscr{E}_m)$ is that

$$\int_{R-F} q(P)\, dP \to \infty, \tag{18}$$

when the cube R, preserving its size, goes to infinity, and the cut F varies arbitrarily while remaining non-essential.

A detailed proof of this theorem may be found in /70/.

We now consider the problem of the spectrum of the operator L generated by operation (1) in some infinite domain $\Omega \subset \mathscr{E}$ with the boundary S and the boundary condition $u = 0$ on this boundary. Here the potential is assumed to be bounded from below, so that $L = L^*$.

In the following considerations, the number of dimensions of the space $\mathscr{E}_m$ is immaterial, and for simplicity we may take $m = 3$.

Let the domain Ω be divided into a finite and infinite part by some plane E. We denote the intersection line of the plane E with the surface S by Γ, and the part of the plane E bounded by this line by R. The operator L induces in a natural manner a regular boundary-value problem in the bounded two-dimensional domain R. If the plane E is given by the equation $z = z_0$, then the induced boundary-value problem is defined by the operation

$$-\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} + q(x, y, z_0)\, u$$

with the boundary condition

$$u(x, y)\big|_{(x, y) \in \Gamma} = 0.$$

Lemma 3 /40/. Let $\lambda(a)$ be the first eigenvalue of the boundary-value problem induced by the operator L in the section of the domain Ω by the plane $z = a$, and let

$$\alpha < \liminf_{a \to \infty} \lambda(a).$$

Under these conditions the part of the spectrum of the operator L which lies to the left of the point α is bounded from below and discrete.

Proof. Let Ω_a denote the unbounded part of the domain Ω cut off from it by the plane $z = a$, and let L_a be the self-adjoint operator generated in $\mathscr{L}_2(\Omega_a)$ by operation (1) with a zero boundary condition on the whole boundary of the domain Ω_a.

We choose the number N so that for $a > N$ we have $\alpha < \lambda(a)$, and show that the self-adjoint operator L_a has no points of the spectrum to the left of α. Assuming the opposite, we obtain a function $u \in \mathfrak{D}_{L_a}$ for which the scalar product $([L_a - \alpha I]u, u)$ is negative, that is,

$$\int_N^\infty dz \iint_{E_z} \{|\nabla u|^2 + (q - \alpha) u^2\}\, dx\, dy < 0, \tag{19}$$

where E_z is the intersection of the domain Ω with the horizontal plane cutting on the Oz axis the intercept z. It follows from (19) that for some $z > N$ we have

$$\iint_{E_z} \left\{ \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 + (q - \alpha) u^2 \right\} dx\, dy < 0$$

and, a fortiori,

$$\int\int_{E_z}\left\{\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+(q-\alpha)u^2\right\}dx\,dy<0. \tag{20}$$

From inequality (20) it follows that the first eigenvalue of the boundary-value problem induced by the operator $L-\alpha I$ in the domain E_z is negative. This contradicts the choice of the number N.

Now it follows from the decomposition principle that the part of the spectrum of the operator L situated to the left of the point α is bounded from below and is discrete.

The particular case $q(P)\equiv 0$ yields the following criterion due to F. Rellich /83 (4)/ for the discreteness of the spectrum of the Laplace operator (see also /40/).

Theorem 4 /83 (4)/. *Let the domain $\Omega\subset\mathbb{E}_m$ contract without limit in the sense that when the hyperplane E is removed to infinity, perpendicularly to some fixed direction, the first eigenvalue of the boundary-value problem induced in $\Omega\cap E$ by the operator $-\Delta$ with zero boundary condition tends to infinity.*

The spectrum in $\mathcal{L}_2(\Omega)$ of the operator $-\Delta$ with zero boundary condition on the boundary of the domain Ω is then bounded from below and discrete.

The proof is completed by referring to Lemma 3.

The criterion established by Theorem 4 is not necessary. The necessary and sufficient condition for the discreteness of the Laplacian spectrum in an infinite domain was obtained by A. M. Molchanov /70/.

For domains with a smooth boundary (e.g., it is sufficient to consider the curvature of the boundary bounded), A.M. Molchamov's criterion maintains that the domain Ω should contain for $r>0$ arbitrarily small only a finite number of non-intersecting spheres of radius r. It is natural to assume that when A. M. Molchanov's condition is satisfied, the eigenvalues of the Laplace operator tend to infinity the faster, the slower the function $n(r)$ increases for $r\to 0$. The function $n(r)$ is equal to the maximum number of non-intersecting spheres of radius r which the domain Ω is capable of enclosing. However, the corresponding asymptotic formulas are unknown. One particular case of a domain satisfying the conditions of Theorem 4 is considered at the end of No. 50. Asymptotic formulas have been obtained for $\Omega=\mathfrak{G}$ and $q(P)\to\infty$ in /95 (2)/ and /61 (2)/.

To conclude this subsection we dwell further on the problem of the application of the decomposition method to the so-called degenerate elliptic equations.

Let

$$l[y]=-\frac{\partial^2 u}{\partial x^2}-\frac{\partial}{\partial y}p(x,y)\frac{\partial u}{\partial y},$$

and (21)

$$p(x,y)\geqslant p_0(y)>0$$

for $y>0$; for $y\to 0$ we may have $\lim p(x,y)=0$. Let the bounded domain lying in the upper half-plane and adjoining the abscissa along the segment $a\leqslant x\leqslant b$ be denoted by Ω. For simplicity, we assume that the part of the boundary complementary to this segment is given by the equation $y=\vartheta(x)$ and that $\vartheta(x)\leqslant 1$. Let the boundary condition $u(x,y)=0$ be satisfied on the boundary of the domain Ω with the exception of the segment $a\leqslant x\leqslant b$. Let L denote the closure of the symmetric operator defined in $\mathcal{L}_2(\Omega)$ by operation (21) on sufficiently smooth functions satisfying the given boundary condition and vanishing near the abscissa.

The operator $L+I$ is obviously positive definite, consequently it is either self-adjoint, or allows only the rigid extension $\mathring{L}+I$. A sufficient condition for the boundedness from below and discreteness of the spectrum $S(\mathring{L})$, by virtue of Theorem 41[bis] of No. 19, is that there exist for any $\mu>0$ a number δ, such that on all the functions $u\in\mathfrak{D}_L$ vanishing near the abscissa and vanishing for $y>\delta$, we have

$$\mu\int_\Omega\int u^2\,dx\,dy<\int_\Omega\int\left\{\left(\frac{\partial u}{\partial x}\right)^2+p\left(\frac{\partial u}{\partial y}\right)^2\right\}dx\,dy. \tag{22}$$

The following theorem of C. G. Mikhlin shows that if $p(x, y)$ does not tend too rapidly to zero when $y \to 0$, then $S(\mathring{L}) = D(\mathring{L})$.

Theorem 5 /69 (2)/. If the minorant $p_0(y)$ of the function $p(x, y)$ satisfies the condition

$$\int_0^1 \frac{y\,dy}{p_0(y)} < \infty, \tag{23}$$

then the spectrum of the non-negative self-adjoint operator $\hat{L}$ is discrete.

Proof. Following M. Sh. Birman, in order to obtain a bound of the form (22) we represent any function $u \in \mathfrak{D}_L$ vanishing near the abscissa and vanishing for $y > \delta$ in the form

$$u(x, y) = -\int_y^\delta \frac{\partial u}{\partial y}\,dy.$$

From this we obtain

$$u^2(x, \beta) \leqslant \int_\beta^\delta p\left(\frac{\partial u}{\partial y}\right)^2 dy \int_\beta^\delta \frac{dt}{p_0(t)} \leqslant$$
$$\leqslant \int_\beta^\delta \left\{\left(\frac{\partial u}{\partial x}\right)^2 + p\left(\frac{\partial u}{\partial y}\right)^2\right\} dy \int_\beta^\delta \frac{dt}{p_0(t)}.$$

Moreover, integrating the two sides of this inequality along the intersection S_β of the domain Ω with the line $y = \beta$, we obtain

$$\int_{S_\beta} u^2(x, \beta)\,dx \leqslant \int_\Omega \left\{\left(\frac{\partial u}{\partial x}\right)^2 + p\left(\frac{\partial u}{\partial y}\right)^2\right\} dx\,dy \int_\beta^\delta \frac{dt}{p_0(t)}.$$

Finally, integrating the two sides of the latter inequality with respect to β from 0 to δ, we obtain

$$\int_{\Omega_\delta} u^2(x, y)\,dx\,dy \leqslant \int_0^\delta d\beta \int_\beta^\delta \frac{dt}{p_0(t)} \int_\Omega \left\{\left(\frac{\partial u}{\partial x}\right)^2 + p\left(\frac{\partial u}{\partial y}\right)^2\right\} dx\,dy, \tag{24}$$

where Ω_δ is the part of the domain Ω situated below the line $y = \delta$.

Comparing relation (24) with inequality (22), we see that for the spectrum $S(\mathring{L})$ to be discrete it is sufficient that the limit relation

$$\lim_{\delta \to 0} \int_0^\delta d\beta \int_\beta^\delta \frac{dt}{p_0(t)} = 0,$$

be satisfied, that is, that the integral

$$\int_0^\delta d\beta \int_\beta^\delta \frac{dt}{p_0(t)} = \int_0^\delta \frac{dt}{p_0(t)} \int_0^t d\beta = \int_0^\delta \frac{t\,dt}{p_0(t)},$$

converges, which is the case by virtue of (23).

48. The negative part of the spectrum and the nodal lines of the solutions of elliptic equations. The results of No. 30 may be generalized to the multi-dimensional case. The following theorem, in particular, corresponds to Kneser's criterion for ordinary differential equations to be oscillatory.

Theorem 6 /31 (10)/. Let

$$\check{q}(r) = \min_{|OP| = r} q(P), \quad \hat{q}(r) = \max_{|OP| = r} q(P).$$

If

$$\liminf_{r \to \infty} r^2 \check{q}(r) > -\frac{(m-2)^2}{4}, \tag{25}$$

then $L = L^*$ and the negative part of the spectrum $S(L)$ consists of a finite number of eigenvalues of finite multiplicity.

If

$$\limsup_{r\to\infty} r^2 \hat{q}(r) < -\frac{(m-2)^2}{4}, \tag{26}$$

then the negative part of any self-adjoint extension $\tilde{L}$ of the operator L is not exhausted by a finite set of eigenvalues of finite multiplicity.

Proof. If condition (25) is satisfied, then the operator L is semi-bounded; consequently, $L = L^*$.

Setting $\alpha = m-3$, $x = r$, in relation (9) of No. 26, we obtain for any finite function $u \in \mathfrak{D}_L$ in any assigned set of angular coordinates the inequality

$$\int_R^\infty r^{m-3}u^2\,dr \leqslant \frac{4}{(m-2)^2}\int_R^\infty r^{m-1}\left(\frac{\partial u}{\partial r}\right)^2 dr.$$

Integrating the two sides with respect to the angular coordinates and replacing $\left(\frac{\partial u}{\partial r}\right)^2$ by $|\nabla u|^2$, we obtain

$$\int_\Omega r^{-2}u^2\,d\omega \leqslant \frac{4}{(m-2)^2}\int_{\Omega_R} |\nabla u|^2\,d\omega$$

or

$$\int_{\Omega_R}\left\{|\nabla u|^2 - \frac{(m-2)^2}{4r^2}u^2\right\} d\omega \geqslant 0, \tag{27}$$

where $m \geqslant 2$.

Using condition (25), we choose the positive numbers δ and R so that for $r \geqslant R$ we have

$$q(P) > -\frac{(m-2)^2-\delta}{4r^2},$$

and we choose a number $k > 0$ satisfying the inequality

$$\frac{(m-2)^2-\delta}{(m-2)^2} < k < 1.$$

To prove the first part of the theorem it is sufficient to establish the inequality

$$\int_{\mathfrak{S}_m}\{|\nabla u|^2 + q(P)|u|^2\}\,d\omega \geqslant 0$$

on the manifold of all the finite functions $u \in \mathfrak{D}_L$ that are orthogonal to some finite-dimensional subspace F in $\mathcal{L}_2(\mathfrak{S}_m)$.

We choose the number μ so that for $|OP| < R$ we have

$$\mu + q(P) + \frac{1-\delta}{4r^2} > 0,$$

and we consider the regular boundary-value problem for $-\Delta$ in the sphere $|OP| \leqslant R$ satisfying $\frac{\partial u}{\partial n} = 0$ on its boundary. Let F denote the linear envelope of the extensions — defined so as to vanish identically outside the

sphere $|OP| = R$ — of all eigenfunctions of this problem corresponding to eigenvalues smaller than $\frac{\mu}{1-k}$.

With this choice of F, we have for all functions $u \in \mathfrak{D}_L$ orthogonal to F,

$$\int\limits_{|OP|<R} \{(1-k)|\nabla u|^2 - \mu |u|^2\}\, d\omega \geqslant 0.$$

On the other hand, from (27) we have

$$\int\limits_{|OP|<R} \left(k|\nabla u|^2 - \frac{(m-2)^2-\delta}{4r^2}|u|^2\right) d\omega + \int\limits_{|OP|>R} \left(k|\nabla u|^2 - \frac{(m-2)^2-\delta}{4r^2}|u|^2\right) d\omega > 0.$$

Adding together the left-hand sides of the last two inequalities, we obtain for any finite function $u \in \mathfrak{D}_L$ orthogonal to F the inequality

$$\int\limits_{|OP|<R} \left\{|\nabla u|^2 - \left(\mu + \frac{(m-2)^2-\delta}{4r^2}\right)|u|^2\right\} d\omega + \int\limits_{|OP|>R} \left\{k|\nabla u|^2 - \frac{(m-2)^2-\delta}{4r^2}|u|^2\right\} d\omega \geqslant 0,$$

from which, due to the choice of the numbers δ, R and μ, we have the required relation

$$\int\limits_{\mathfrak{S}_m} \{|\nabla u|^2 + q(P)|u|^2\}\, d\omega \geqslant 0.$$

To prove the second part of the theorem, we write the expression for the quadratic functional

$$\hat{\Phi}[u] = \int\limits_{\mathfrak{S}_m} \{|\nabla u|^2 + \hat{q}|u|^2\}\, d\omega$$

in terms of spherical coordinates, and change the variables according to formula (10). The function $v(P) = v(r)$ is assumed to be R-finite. We thus obtain

$$\hat{\Phi}[u] = C_m \int\limits_R^\infty \left\{\left|\frac{\partial v}{\partial r}\right|^2 + \left[\hat{q}(r) + \frac{(m-2)^2-1}{4r^2}\right] v^2\right\} dr, \tag{28}$$

where the constant C_m is the surface area of a unit sphere in $\mathfrak{S}_m$.

The right-hand side of (28) defines the quadratic functional

$$F[v] = \int\limits_0^\infty \left\{\left|\frac{\partial v}{\partial r}\right|^2 + \tilde{q}(r)|v|^2\right\} dr$$

in $\mathscr{L}_2(0, \infty)$, where

$$\tilde{q}(r) = \hat{q}(r) + \frac{(m-2)^2-1}{4r^2}. \tag{29}$$

From (26), we have for large r the following relation

$$\tilde{q}(r) < -\frac{1+\delta}{4r^2}.$$

which is the condition for the oscillatoriness of the one-dimensional operation

$$-\frac{d^2v}{dr^2}+\tilde{q}(r)v$$

(Theorem 9 of No. 30). There exists therefore an infinite sequence $\{v_n(r)\}$ of finite functions with disjoint supports, for which $F[v_n]<0$. Setting

$$u_n(P)=r^{\frac{1-m}{2}}v_n(r),$$

we obtain

$$(Lu_n,\ u_n)\leqslant\int\limits_{\mathfrak{E}_m}\{|\nabla u_n|^2+\hat{q}|u_n|^2\}\,d\omega=C_mF[v_n]<0.$$

It therefore follows that the negative part of any self-adjoint extension $\tilde{L}$ of the operator L cannot be exhausted by a finite set of eigenvalues of finite multiplicity. The theorem is thus proved.

From the second part of the proof it follows that all the one-dimensional criteria for the infinity of the set of points of the negative part of the spectrum (Theorems 11-14 of No. 30) can be carried over to the multi-dimensional case, providing we set $n=2$ in them and replace the potential $q(x)$ by the function $\tilde{q}(r)$ defined by formula (29).

If the function $u(P)$ is a solution of the equation

$$-\Delta u+q(P)u=0 \tag{30}$$

in a bounded (not necessarily simply connected) domain $\omega\subset\mathfrak{E}_m$, is continuous up to the boundary Γ of this domain, and is equal to zero on this boundary, then the boundary Γ is called the nodal contour of the solution $u(P)$.

As in the one-dimensional case, a relation exists between the character of the negative part of the spectrum and the nodal contours of the solution of equation (30). We call (30) a non-oscillatory equation, if for some R in the domain $|OP|>R$ its solutions do not have nodal contours. Conversely, when for any R there exist solutions with nodal contours in the domain $|OP|>R$, we call (30) an oscillatory equation. The simplest example of a non-oscillatory equation is obtained in the case of a finite potential, since then in some domain $|OP|>R$ the solutions of equation (30) are harmonic functions.

Theorem 7. Under condition (25), equation (30) is non-oscillatory, and under condition (26) it is oscillatory.

Proof. Without loss of generality we take $m=2$.

If condition (25) is satisfied, the negative part of the spectrum of the self-adjoint operator L contains only a finite number of eigenvalues of finite multiplicity, according to Theorem 6.

Suppose the statement regarding the non-oscillatory character were false. Then there would exist an infinite sequence of finite functions $\{\psi_k(P)\}$ with disjoint supports ω_k, bounded by the nodal contours Γ_k, for which

$$\Delta\psi_k-q(P)\psi_k=0\qquad(P\in\omega_k;\ k=1,\ 2,\ \ldots). \tag{31}$$

Equality (31) implies that $\lambda_k=0$ and $\psi_k(P)$ are, respectively, an eigenvalue and an eigenfunction of the boundary-value problem defined in $\mathfrak{D}_2(\omega_k)$ by operation (1) with the boundary condition $u(P)|_{P\in\Gamma_k}=0$.

Together with this problem we consider the boundary-value problem defined by the same operation (1) in $\mathfrak{L}_2(\tilde{\omega}_k)$ with the boundary condition $u(P)_{P\in\tilde{\Gamma}_k}=0$. Here $\tilde{\Gamma}_k$ is the complete boundary of some domain $\tilde{\omega}_k\supset\omega_k$ differing from ω_k by so little that for $j\neq k$ we have $\tilde{\omega}_j\cap\tilde{\omega}_k\neq\emptyset$ $(j, k=1, 2, \ldots)$.

On the basis of the theorem formulated by R. Courant /54/ and proved by Yu. I. Lyubich /64 (1)/, on passing from ω_k to the domain $\tilde{\omega}_k\supset\omega_k$, each eigenvalue strictly decreases. We therefore have an infinite system $\{\tilde{\psi}_k(P)\}$ of finite functions with disjoint supports $\tilde{\omega}_k$, satisfying the equation

$$-\Delta\tilde{\psi}_k+q(P)\tilde{\psi}_k=\lambda_k\tilde{\psi}_k \qquad (P\in\tilde{\omega}_k),$$

where

$$\lambda_k<0 \qquad (k=1, 2, \ldots),$$

so that we have

$$(l[\tilde{\psi}_k],\ \tilde{\psi}_k)<0.$$

Smoothing the function $\tilde{\psi}_k(P)$ over the nodal contours $\tilde{\Gamma}_k$ by means of Lemma 13 of No. 19, we obtain an infinite system of finite functions $\{\varphi_k(P)\}$ with disjoint supports $\tilde{\omega}_k$, belonging to $\mathfrak{D}_L$ and satisfying the inequality

$$(L\varphi_k,\ \varphi_k)<0 \qquad (k=1, 2, \ldots).$$

It therefore follows that the set of points of the negative part of the spectrum $S(L)$ is infinite. This contradicts condition (25).

Suppose inequality (26) is given. As was shown in the proof of the second part of Theorem 6, it follows from this inequality that for sufficiently large R there exists an R-finite function $u(P)\in\mathfrak{D}_L$, for which

$$(Lu,\ u)<0. \tag{32}$$

If $u(P)=0$ for $|OP|>\rho$, then from inequality (32) it follows that the first eigenvalue, $\lambda(\rho)$, of the boundary value problem defined by operation (1) in the annulus $R\leqslant|OP|\leqslant\rho$ with zero boundary condition on its boundary is negative. Now, when ρ decreases to $\rho=R$ the eigenvalue $\lambda(\rho)$ increases continuously /54/ up to $+\infty$, thus for some $\rho=R_1$ we have $\lambda(\rho)=0$. The corresponding eigenfunction is then the required solution, having a nodal contour in the domain $|OP|>R$.

The first part of the proof establishes slightly more than indicated in the formulation of the theorem; namely, it has been shown that the non-oscillatory character follows from the finiteness of the negative part of the spectrum of the operator $\tilde{L}$.

A similar strengthening of the second part of the theorem for $m\geqslant 2$ without additional assumptions is impossible: from the infiniteness of the set of points of the negative part of the spectrum of some self-adjoint extension $\tilde{L}$ of the operator L it does not follow that equation (30) is oscillatory. However, the problem of strengthening the second part of the theorem under the additional assumption that the set of points of the negative part of the spectrum of any self-adjoint extension $\tilde{L}$ of the operator L is infinite remains open. This problem remains open even with the additional requirement $L=L^*$(see the end of No. 19).

49. The character of the continuous part of the spectrum as related to the behavior of the potential and to the form of the infinite domain. For the convenience of the formulation of the theorems of the present subsection we adopt the following classification of the domains of the space $\mathfrak{E}_m$

Let $\dot{\Omega}$ denote any (connected or unconnected) domain of the space. We call the domain $\dot{\Omega}$ quasi-conical if it contains spheres of any radius. If the domain $\dot{\Omega}$ is not quasi-conical, but contains an infinite set of identical non-intersecting spheres, it is called quasi-cylindrical.

In particular, this set of spheres itself is a quasi-cylindrical domain. Finally, if the domain $\dot{\Omega}$ does not contain such a set of spheres, it is called quasi-bounded. Obviously, any domain $\dot{\Omega} \subset \mathfrak{G}_m$ belongs to one of the above types.

The criteria given below refer to the continuous part of the spectrum of any self-adjoint extension $\tilde{L}$ of the symmetric operator L. The operator $\tilde{L}$ may be generated by operation (1) in the whole space $\mathfrak{G}_m$ or in an arbitrary part of it, Ω, containing the domain $\dot{\Omega}$, with any boundary conditions on the boundary of this part. No restrictions, apart from continuous differentiability, are imposed on the behavior of the potential $q(P)$ outside $\dot{\Omega}$

Without loss of generality we take $m=3$ and use the functions

$$u_j(P) = u_j(P;\ \lambda,\ \eta) = \varphi(x-\alpha_j;\ \lambda,\ \eta)\,\varphi(y-\beta_j;\ 0, \eta)\,\varphi(z-\gamma_j;\ 0,\ \eta); \tag{33}$$

as test systems of finite functions with disjoint supports. The function $\varphi(x;\ \lambda,\ \eta)$ is defined here by equality (73) of No. 31, so that for all natural numbers j we have

$$\|u_j(P;\ \lambda,\ \eta\| = 1, \quad \max_{P \in \mathfrak{G}} |u_j(P;\ \lambda,\ \eta)| = \varepsilon^{\frac{3}{2}} \tag{34}$$

and

$$\|\Delta u_j + \lambda u_j\| < \varepsilon; \tag{35}$$

the functions (33) obviously belong to the domain $\mathfrak{D}_{\tilde{L}}$ of any extension of the operator L.

Theorem 8 /31 (4)/. If the domain $\dot{\Omega}$ is quasi-conical and

$$\omega = \lim_{|OP| \to \infty} \sup q(P) - \lim_{|OP| \to \infty} \inf q(P) \qquad (P \in \dot{\Omega}),$$

then for large λ we have

$$[\lambda,\ \lambda + \omega] \cap C(\tilde{L}) \neq \emptyset.$$

In particular, if

$$\lim_{|OP| \to \infty} q(P) = 0 \qquad (P \in \dot{\Omega}),$$

then

$$[0,\ \infty) \subset C(\tilde{L}).$$

Proof. Since the domain Ω is quasi-conical, then for any $\eta > 0$ it is possible to choose a point $(\alpha_j, \beta_j, \gamma_j)$ in (33), such that for all natural numbers j the supports of the finite functions $u_j(P)$ lie inside $\dot{\Omega}$ and do not intersect. Moreover, for a given $\delta > 0$ we choose a number $N = N_\delta$, such that for $P \in \dot{\Omega}$ we have

$$\sup_{|OP| > N} q(P) - \inf_{|OP| > N} q(P) < \omega + \delta.$$

Assuming

$$\lambda > \frac{1}{2}\left\{ \lim_{|OP| \to \infty} \sup q(P) - \lim_{|OP| \to \infty} \inf q(P) \right\} \qquad (P \in \dot{\Omega}),$$

we obtain, as in the proof of Theorems 17 and 19 of No. 31, that the inequality

$$\|Lu_j - \lambda u_j\| \leqslant (\eta + \omega + \delta)\|u_j\|.$$

is satisfied for all but a finite number of functions of the system (33). Since the quantities $\eta > 0$ and $\delta > 0$ are arbitrary, the first part of the theorem follows from the above inequality. The second part of the theorem follows directly from the first part.

Theorem 9 /31 (4)/. Let $\dot{\Omega}$ be any quasi-conical domain. If the condition

$$\int_{M_\delta} |q(P)|^2 d\omega < \infty,$$

where M_δ is the set of the points $P \in \dot{\Omega}$, for which $|q(P)| \geqslant \delta$, is satisfied in $\dot{\Omega}$ for every positive δ, then the positive part of the spectrum of the operator $\tilde{L}$ covers the entire semi-axis $\lambda \geqslant 0$.

Proof. Choosing for a fixed $\lambda > 0$ and an arbitrary $\eta > 0$ a point $(\alpha_j, \beta_j, \gamma_j)$ in (33), such that for all natural numbers j the supports of the finite functions $u_j(P)$ lie inside $\dot{\Omega}$ and do not intersect, we obtain, as in No. 31,

$$\|\tilde{L}u_j - \lambda u_j\| \leqslant \eta \|u_j\| + \|qu_j\| \leqslant \eta \|u_j\| +$$

$$+ \left\{ \int_{\Omega - M_\delta} |q(P) u_j(P)|^2 d\omega \right\}^{\frac{1}{2}} + \left\{ \int_{M_\delta} |q(P) u_j(P)|^2 d\omega \right\}^{\frac{1}{2}} \leqslant \rho(\varepsilon, \delta) \|u_j\|,$$

where

$$\rho(\varepsilon, \delta) = \eta + \delta + \varepsilon^{\frac{3}{2}} \left\{ \int_{M_\delta} |q(P)|^2 d\omega \right\}^{\frac{1}{2}}.$$

Since the quantity $\rho(\varepsilon, \delta)$ may be made arbitrarily small together with $\varepsilon > 0$ and $\delta > 0$, then $\lambda \in C(\tilde{L})$. This proves the theorem.

Theorem 10. If the relation

$$\lim_{j \to \infty} \frac{1}{|Q_j|} \int_{Q_j} |q(P)|^2 d\omega = 0,$$

for some sequence of cubes Q_j with sides Δ_j of length $|\Delta_j| = \sqrt[3]{|Q_j|}$ increasing without limit is satisfied, then the positive part of the spectrum of the operator $\tilde{L}$ covers the entire semi-axis $\lambda \geqslant 0$.

Proof. Setting

$$\varepsilon = \varepsilon_j = \frac{1}{2|\Delta_j|}$$

in formula (73) of No. 31, and taking the centers of the cubes Q_j as the points $(\alpha_j, \beta_j, \gamma_j)$ in (33), we obtain for the normalized functions $u_j(P)$, as in the proof of Theorem 22 of No. 31, the inequality

$$\|Lu_j - \lambda u_j\| \leqslant \eta_j \|u_j\| + \|qu_j\| \leqslant \eta_j + \left\{ \frac{1}{8|Q_j|} \int_{Q_j} |q(P)|^2 d\omega \right\}^{\frac{1}{2}},$$

so that

$$\lim_{j \to \infty} \|\tilde{L}u_j - \lambda u_j\| = 0,$$

leading to the required relation $\lambda \in C(\tilde{L})$.

Theorem 11 /31 (4)/. If in some quasi-cylindrical domain $\dot{\Omega}$ the potential $q(P)$ is bounded, then for large λ the length of the gaps in $C(\tilde{L})$ centered at the point λ does not exceed $O(\sqrt{\lambda})$.

Proof. Since the domain $\dot{\Omega}$ is quasi-cylindrical, then for a certain ε it is possible to choose the points $(\alpha_j, \beta_j, \gamma_j)$ so that for any natural number j the supports of the finite functions (33) lie inside $\dot{\Omega}$ and do not intersect.

By means of formulas (73) of No. 31 and (33), the inequality

$$\|Lu_j - \lambda u_j\| < (\alpha\sqrt{\lambda} + \beta)\|u_j\|$$

is easily obtained for any natural number j for $\lambda > \lambda_0$. The theorem is thus proved. We note that the numbers λ_0, α, and β do not depend on the choice of the extension.

Remark. The use of the same test system of functions (33) in the case of an ordinary cylindrical domain with any generator leads to the replacement of $O(\sqrt{\lambda})$ by $O(1)$.

As in the one-dimensional case, the following alternatives are possible for the operator $\dot{L}$ in $\mathcal{L}_2(\mathfrak{G}_m)$

Theorem 12. If the operator L, defined by operation (1) in $\mathcal{L}_2(\mathfrak{G}_m)$, is stably positive, then either the self-adjoint operator L has no continuous spectrum, or its continuous spectrum extends to infinity and has gaps of length not larger than $O(\sqrt{\lambda})$.

Given the same conditions, either the point $\lambda = 0$ does not belong to $C(L)$, or $[0, \infty) \subset C(L)$.

The proof of the two statements does not differ from that given in No. 32.

The following theorems of this subsection deal with the spectrum of the Laplace operator. By $-\tilde{\Delta}$ we shall denote the self-adjoint operator in $\mathcal{L}_2(\Omega)$, defined by the operation $-\Delta$ with zero boundary condition on the boundary of the domain Ω.

In the case of the Laplacian we have the following alternative.

Theorem 13 /31 (4)/. Whatever the domain $\Omega \subset \mathfrak{G}_m$ may be, if the boundary is sufficiently smooth then the continuous part of the spectrum of the operator $-\tilde{\Delta}$ is either absent, or it extends to infinity and has gaps of length not larger than $O(\sqrt{\lambda})$.

Proof. If the domain Ω is quasi-conical or quasi-cylindrical, then, by virtue of Theorems 9 and 12, the continuous spectrum will extend to infinity, either covering completely the semi-axis $\lambda \geqslant 0$ or having gaps of length not larger than $O(\sqrt{\lambda})$.

If the domain Ω is neither quasi-conical nor quasi-cylindrical, it is quasi-bounded. But then, if the boundary is sufficiently smooth, the spectrum of the operator $-\tilde{\Delta}$ will be discrete, according to A. M. Molchanov's theorem /70/ previously mentioned in connection with Theorem 4. The theorem is thus proved.

Let $\Omega \subset \mathfrak{G}$ be a semi-infinite cylinder with a bounded base lying in the plane $z = 0$ and having generators parallel to the z axis. In this simple case, the spectrum $S(-\tilde{\Delta})$ is easily found by separation of variables. It is found that $D(-\tilde{\Delta}) = 0$ and $S(-\tilde{\Delta}) = C(-\tilde{\Delta}) = [0, \infty)$, where α is the first eigenvalue of the boundary-value problem induced in the section $z = z_0$.

The same result is obtained when the zero boundary condition on the cylinder base is replaced by the condition $\frac{\partial u}{\partial n} = 0$.

If we pass, by means of a finite deformation, from the cylinder Ω to some domain Ω', then the continuous part of the spectrum does not vary, but eigenvalues may appear outside $C(-\tilde{\Delta})$ and in $C(-\tilde{\Delta})$. By virtue of Lemma 1 the number of eigenvalues outside $C(-\tilde{\Delta})$ is finite. Of what type, then, is the set of eigenvalues appearing in $C(-\tilde{\Delta})$? The answer to this question is given by the following theorem due to Jones.

Theorem 14 /40/. The discrete part of the spectrum of the operator $-\tilde{\Delta}$ in the domain obtained from a cylinder by means of a finite deformation consists of isolated eigenvalues of finite multiplicity with the single possible limit point $\lambda = +\infty$.

Proof. It is obvious that the operator $-\tilde{\Delta}$ is not negative and thus its discrete spectrum $D(-\tilde{\Delta})$ is situated on the semi-axis $\lambda \geqslant 0$.

Let $\alpha_1 = \alpha, \alpha_2, \alpha_3, \ldots$ denote the successive eigenvalues of the boundary-value problem induced by the operator $-\tilde{\Delta}$ in the cross section of the cylindrical part of the given domain Ω. Without loss of generality, we consider the spectrum of this induced operator to be simple, so that

$$0 < \alpha_1 < \alpha_2 < \ldots$$

Let the corresponding eigenfunctions be

$$w_1(x, y), \quad w_2(x, y), \quad \ldots$$

We choose a number a so large that the plane $z = a$ divides Ω into a bounded part ω and an infinite cylindrical part Ω_a.

Let

$$\lambda_1' \leqslant \lambda_2' \leqslant \lambda_3' \leqslant \ldots$$

denote the successive eigenvalues of the Laplacian in ω with the boundary condition $\frac{\partial u}{\partial z} = 0$ for $z = a$ and the zero boundary condition $u = 0$ on the remaining part of the boundary of the domain ω. The corresponding eigenfunctions are denoted by

$$\psi_1(P), \psi_2(P), \psi_3(P), \quad \ldots \tag{36}$$

We first show that to the left of the point α_1 the spectrum of the operator $-\tilde{\Delta}$ in $\mathscr{D}_2(\Omega)$ is exhausted by a finite number of eigenvalues of finite multiplicity. For this purpose we note that the operator $-\tilde{\Delta}_a$ in $\mathscr{D}_2(\Omega_a)$, with the boundary condition $\frac{\partial u}{\partial z} = 0$ for $z = a$ and zero boundary conditions on the remaining part of the boundary of the domain Ω_a, does not have any points of the spectrum to the left of α_1.

In fact, assuming the opposite, we can find a function $u \in \mathfrak{D}_{\tilde{\Delta}_a}$, for which

$$\int_a^\infty dz \int_{E_z} \int \{ |\nabla u|^2 - \alpha_1 |u|^2 \} \, dx \, dy < 0,$$

where E_z is the intersection of the domain Ω with the horizontal plane having on the Oz axis the intercept z. We then have, for some $z > a$,

$$\int_{E_z} \int \left\{ \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 - \alpha_1 u^2 \right\} dx \, dy < 0$$

and, a fortiori,

$$\int_{E_z} \int \left\{ \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 - \alpha_1 u^2 \right\} dx \, dy < 0.$$

This is impossible, since α_1 is the first eigenvalue of the induced boundary-value problem.

Thus, α_1 is the first point of the spectrum of the operator $-\tilde{\Delta}_a$ and, using Lemma 1, we conclude that to the left of the point α_1 the spectrum of the operator $-\tilde{\Delta}$ in $\mathscr{L}_2(\Omega)$ is exhausted by a finite number of eigenvalues of finite multiplicity.

To prove the theorem it remains to show that in each of the intervals

$$(\alpha_1, \alpha_2], \quad (\alpha_2, \alpha_3], \quad \ldots \tag{37}$$

the number of eigenvalues of finite multiplicity belonging to the given operator $-\tilde{\Delta}$ in $\mathscr{L}_2(\Omega)$ is finite.

Let $v(x, y, z)$ be an eigenfunction of the operator $-\tilde{\Delta}$ in $\mathscr{L}_2(\Omega)$. For a fixed $z > a$, this function can be represented by the orthogonal series

$$v(x, y, z) = \sum_{j=1}^{\infty} c_j(z)\, w_j(x, y), \tag{38}$$

so that for any n

$$\sum_{j=1}^{n} |c_j(z)|^2 \leqslant \int\int_{E_z} |v(x, y, z)|^2\, dx\, dy$$

and consequently

$$\sum_{j=1}^{n} \int_a^{\infty} |c_j(z)|^2\, dz \leqslant \|v\|_2^2. \tag{39}$$

On the other hand, substituting the series (38) in the relation

$$\Delta u + \lambda u = 0, \tag{40}$$

we obtain the ordinary differential equations for the coefficients of this series

$$c_j''(z) + (\lambda - \alpha_j)\, c_j(z) = 0 \qquad (z > a;\ j = 1, 2, \ldots). \tag{41}$$

If the eigenvalue λ of the operator $-\tilde{\Delta}$ in $\mathscr{L}_2(\Omega)$ belongs to the first of the intervals of the system (37), then $\lambda > \alpha_1$. However, it then follows from inequality (39) and the differential equation (41) that

$$c_1(z) = 0 \qquad (z > a). \tag{42}$$

Now it is not difficult to show that in the interval $(\alpha_1, \alpha_2]$ there can be no more than n' eigenvalues of the operator $-\tilde{\Delta}$ in $\mathscr{L}_2(\Omega)$, where the number n' is determined by the inequality

$$\lambda'_{n'} \leqslant \alpha_2 < \lambda'_{n'+1}.$$

In fact, assuming the opposite, we can form a linear combination, $\varphi(P)$, of eigenfunctions of the operator $-\tilde{\Delta}$ in $\mathscr{L}_2(\Omega)$ which correspond to the eigenvalues of the interval $(\alpha_1, \alpha_2]$, such that the linear combination is orthogonal in $\mathscr{L}_2(\omega)$ to the first n' functions of the system (36). For the function $\varphi(P)$ we obviously have

$$\int_{\Omega} \{|\nabla\varphi|^2 - \alpha_2 |\varphi|^2\}\, d\omega \leqslant 0,$$

that is

$$\int_{\omega} \{|\nabla\varphi|^2 - \alpha_2 |\varphi|^2\}\, d\omega + \int_{\Omega_a} \{|\nabla\varphi|^2 - \alpha_2 |\varphi|^2\}\, d\omega \leqslant 0. \tag{43}$$

But from the orthogonality of $\varphi(P)$ to the eigenfunctions $\{\psi_j(P)\}_{j=1}^{n'}$ of the regular problem it follows that the first term in (43) is positive. We therefore have

$$\int_a^\infty dz \iint_{E_z} \{|\nabla\varphi|^2 - \alpha_2|\varphi|^2\}\, dx\, dy < 0,$$

consequently, for some $z > a$

$$\iint_{E_z} \left\{\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2 - \alpha_2|\varphi|^2\right\} dx\, dy < 0. \tag{44}$$

On the other hand, all the eigenfunctions of the operator $-\tilde{\Delta}$ in $\mathscr{L}_2(\Omega)$ that belong to the linear combination $\varphi(P)$ should by virtue of (42) be orthogonal to $w_1(x, y)$ in $\mathscr{L}_2(E_z)$, so that

$$\iint_{E_z} \varphi(x, y, z)\,\overline{w_1(x, y)}\, dx\, dy = 0. \tag{45}$$

By virtue of the extremal properties of the eigenvalues it follows from relations (44) and (45) that in the induced problem an eigenvalue should exist inside the interval (α_1, α_2). This contradiction shows the inconsistency of the assumption of the existence in the interval $(\alpha_1, \alpha_2]$ of more than n' eigenvalues of the operator $-\tilde{\Delta}$ in $\mathscr{L}_2(\Omega)$.

Passing to the second interval of the system (37) in exactly the same way, using instead of the single relation (42) the two following from (39) and (40)

$$c_1(z) \equiv c_2(z) \equiv 0 \qquad (z > a),$$

we obtain that in the interval $(\alpha_2, \alpha_3]$ there may be no more than n_2' eigenvalues of the operator $-\tilde{\Delta}$, where

$$\lambda'_{n_2'} \leqslant \alpha_3 < \lambda'_{n_2'+1}.$$

Passing thus from each interval of the system (37) to the next, we obtain the complete proof of the theorem.

The following theorem, due to F. Rellich, shows that under certain restrictions on the form of the boundary of the quasi-conical domain the continuous part of the spectrum of the Laplacian contains no eigenvalues. The part of the space, for example, enclosed inside an elliptical paraboloid, is one of these domains.

Theorem 15 /83 (2)/. If the external normal at each point P of the boundary S of a quasi-conical domain Ω, situated in the half-space $z>a$ $(a>-\infty)$, makes an angle $\beta(P) \geqslant \frac{\pi}{2}$ with the positive direction of the z axis, then there are no eigenvalues of the operator $-\tilde{\Delta}$ on the continuous part of the spectrum $C(-\tilde{\Delta}) = [0, \infty)$.

Proof. It follows from the non-negativeness of the operator $-\tilde{\Delta}$ that $S(-\tilde{\Delta}) \subset [0, \infty)$, and it follows from Theorem 8 that $[0, \infty) \subset C(-\tilde{\Delta})$. Thus

$$S(-\tilde{\Delta}) = C(-\tilde{\Delta}) = [0, \infty).$$

In order to prove the theorem it is sufficient to show that for $\lambda \geqslant 0$ equation (43) does not have a solution $u(P)$ in $\mathscr{L}_2(\Omega)$ which is equal to zero on S and not identically zero in Ω.

We assume that for any ζ the part of the domain Ω lying between the planes $z = a$ and $z = \zeta$ is bounded, and we denote it by ω_ζ. Furthermore,

we denote by σ_ζ that part of the plane $z=\zeta$, which lies in Ω, and the remaining part of the boundary of ω_ζ we denote by S_ζ.

We now prove that if the function $u(P)$ satisfies equation (43) for $\lambda \geqslant 0$, vanishes on S, and is not identically zero, then there exists a number $\mu > 0$ and a sequence $\zeta_n \to \infty$ such that

$$\int\limits_{\omega_{\zeta_n}} |u(P)|^2 \, d\omega \geqslant \mu \zeta_n ; \tag{46}$$

consequently, $u \overline{\in} \mathscr{L}_2(\Omega)$.

In the proof the function $u(P)$ may be considered to be real, since if $u_1 = \operatorname{Re} u$, $u_2 = \operatorname{Im} u$, and inequality (46) is satisfied for u_1 and u_2 when $\mu = \mu_1$, $\zeta_n = \zeta_n^{(1)}$ and $\mu = \mu_2$, $\zeta_n = \zeta_n^{(2)}$ respectively, then for $\mu = \min\{\mu_1, \mu_2\}$ we have

$$\int\limits_{\omega_{\zeta_n}^{(1)}} |u_1(P)|^2 \, d\omega + \int\limits_{\omega_{\zeta_n}^{(2)}} |u_2(P)|^2 \, d\omega \geqslant \mu (\zeta_n^{(1)} + \zeta_n^{(2)}),$$

hence, (46) follows for $\zeta_n = \zeta_n^{(1)} + \zeta_n^{(2)}$.

Assuming the solution $u(P)$ to be real, we have

$$0 = (\Delta u + \lambda u) \frac{\partial u}{\partial z} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} \cdot \frac{\partial u}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y} \cdot \frac{\partial u}{\partial z}\right) +$$

$$+ \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z} \cdot \frac{\partial u}{\partial z}\right) - \frac{1}{2}\frac{\partial}{\partial z}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2\right] + \frac{\lambda}{2}\frac{\partial u^2}{\partial z},$$

and by integrating on [the domain] ω_ζ we obtain

$$\int\limits_{\omega_\zeta} \left\{ \operatorname{div}\left[\frac{\partial u}{\partial z} \nabla u\right] - \frac{1}{2}\frac{\partial}{\partial z}|\nabla u|^2 + \frac{\lambda}{2}\frac{\partial u^2}{\partial z} \right\} d\omega = 0$$

or, denoting differentiation along the external normal to S by $\frac{\partial}{\partial n}$,

$$\int\limits_{S_\zeta + \sigma_\zeta} \left\{ \frac{\partial u}{\partial z}\frac{\partial u}{\partial n} - \frac{1}{2}|\nabla u|^2 \frac{\partial z}{\partial n} + \frac{\lambda}{2} u^2 \frac{\partial z}{\partial n} \right\} d\sigma = 0. \tag{47}$$

Moreover, since $u(P) = 0$ for $P \in S$, we then have on S that $\nabla u = \frac{\partial u}{\partial n} n$, where n is the versor of the external normal to S. Relation (47) therefore takes the form

$$\int\limits_{S_\zeta} \left(\frac{\partial u}{\partial n}\right)^2 \frac{\partial z}{\partial n} \, d\sigma + \int\limits_{\sigma_\zeta} \left\{ \left(\frac{\partial u}{\partial z}\right)^2 - \left(\frac{\partial u}{\partial x}\right)^2 - \left(\frac{\partial u}{\partial y}\right)^2 + \lambda u^2 \right\} d\sigma = 0. \tag{48}$$

On the other hand, since $\frac{\partial z}{\partial n} \leqslant 0$ on S, then for $\zeta \geqslant \zeta_0$ we have

$$-\int\limits_{S_\zeta} \left(\frac{\partial u}{\partial n}\right)^2 \frac{\partial z}{\partial n} \, d\sigma \geqslant -\int\limits_{S_{\zeta_0}} \left(\frac{\partial u}{\partial n}\right)^2 \frac{\partial z}{\partial n} \, d\sigma = \rho \geqslant 0. \tag{49}$$

We choose the number ζ_0 so that we have $\rho > 0$. This is possible, since the domain Ω is not a cylinder, and consequently in the neighborhood of some point on S we have $\frac{\partial z}{\partial n} < 0$. In this neighborhood we do not have $\frac{\partial u}{\partial n} = 0$ everywhere, since in view of the fact that $u = 0$ on S we would then have, in accordance with the uniqueness theorem /59/, that $u(P) \equiv 0 \, (P \in \Omega)$.

Thus for $\zeta \geqslant \zeta_0$ we have, by virtue of (48) and (49),

$$\int\limits_{\sigma_\zeta} \left[\left(\frac{\partial u}{\partial z}\right)^2 - \left(\frac{\partial u}{\partial x}\right)^2 - \left(\frac{\partial u}{\partial y}\right)^2 + \lambda u^2\right] d\sigma \geqslant \rho > 0$$

and, a fortiori,

$$\int\limits_{\sigma_\zeta} \{|\nabla u|^2 + \lambda u^2\} \, d\sigma \geqslant \rho > 0,$$

whence

$$\int\limits_{\omega_\zeta} \{|\nabla u|^2 + \lambda u^2\} \, d\omega \geqslant \rho(\zeta - \zeta_0). \tag{50}$$

In order to derive inequality (46) from relation (50), we introduce the function

$$J(\zeta)=\int\limits_{\sigma_\zeta} u^2\,d\sigma$$

and distinguish between the case when the function monotonically increases for $\zeta\to\infty$, and the opposite case.

In the first case we have for all ζ and a certain $\mu>0$

$$J(\zeta)>\mu,$$

whence

$$\int\limits_{\omega_\zeta} u^2\,d\omega>\mu\zeta+C.$$

This implies that inequality (46) is fulfilled.

In the second case there exists a sequence $\zeta_n\to\infty$, such that

$$J'(\zeta_n)=2\int\limits_{\sigma_\zeta} u\frac{\partial u}{\partial z}\,d\sigma\leqslant 0.$$

But then from the identity

$$0=\int\limits_{\omega_\zeta}(\Delta u+\lambda u)\,u\,d\omega=\int\limits_{\sigma_\zeta} u\frac{\partial u}{\partial z}\,d\sigma-\int\limits_{\omega_\zeta}|\nabla u|^2\,d\omega+\lambda\int\limits_{\omega_\zeta} u^2\,d\omega$$

it follows that

$$\int\limits_{\omega_{\zeta_n}}|\nabla u|^2\,d\omega\leqslant\lambda\int\limits_{\omega_{\zeta_n}} u^2\,d\omega. \tag{50}$$

and now the required inequality (46) follows from (50).

T. Kato has succeeded in finding an estimate, similar to inequality (46), for the case of the Schroedinger operator with the potential $q(P)=o(|OP|)$ thus proving the absence of eigenvalues in the continuous part of the spectrum of such operators (see No. 65).

50. On the discrete part of the spectrum of the Laplace operator in limit-cylindrical domains. Let E_0 be some star-like domain, with respect to the origin of the coordinates of the plane $z=0$. Let its boundary be given by

$$r=r(\varphi)\qquad(0\leqslant\varphi<2\pi). \tag{51}$$

The domain $\Omega\subset\mathfrak{S}$, bounded by the surface

$$r=r(\varphi)(1+\delta(z))\quad(0\leqslant\varphi<2\pi,\ \delta(z)>-1,\ 0\leqslant z<\infty) \tag{52}$$

(in cylindrical coordinates) and by the definite part of the plane $z=0$ cut by this surface, is said to be limit-cylindrical if

$$\lim_{z\to\infty}\delta(z)=0. \tag{53}$$

Let $-\tilde{\Delta}$ denote the self-adjoint operator defined in $\mathfrak{L}_2(\Omega)$ by the operation $-\Delta$ with zero boundary condition on the boundary of the domain Ω. Furthermore, let α denote the first eigenvalue of the boundary-value problem defined by the two-dimensional operation $-\Delta$ in the plane domain E_0 with zero boundary condition on its boundary (51).

It is easy to show that the number α is the same as the lower bound α^* of the continuous part of the spectrum of the operator $-\tilde{\Delta}$ in the limit-cylindrical domain Ω.

Indeed, according to Lemma 3, to the left of the point α the spectrum $S(-\tilde{\Delta})$ is discrete, that is, $\alpha \leqslant \alpha^*$. On the other hand, it is possible to find for any $\varepsilon > 0$ a number $h (0 < h < 1)$ such that for $\delta(z) \equiv -h$ the first eigenvalue of the problem induced in the section $z = 0$ is equal to $\alpha + \varepsilon$. The limit-cylindrical domain is then transformed into a cylinder, and, by virtue of (53), for a certain $z_0 > 0$ the part C_0 of this cylinder which is situated above the plane $z = z_0$ lies inside the given limit-cylindrical domain Ω. The number $\alpha + \varepsilon$ is obviously the first point of the continuous part of the spectrum of the operator $-\tilde{\Delta}$ in $\mathscr{L}_2(C_0)$ (see page 162). By virtue of Lemma 2, we then have $\alpha^* \leqslant \alpha + \varepsilon$, and by letting ε tend to 0 we obtain $\alpha^* \leqslant \alpha$.

Now, using Lemma 1, we conclude that under any finite deformation of the cylindrical domain the first point α of the continuous part of the spectrum $-\tilde{\Delta}$ is not a limit point for the eigenvalues situated on its left. It is therefore natural to assume that if the function $\delta(z)$ tends sufficiently rapidly to zero, then to the left of the point α there is only a finite set of points of the spectrum $S(-\tilde{\Delta})$.

On the other hand, if in (2) we set $\delta(z) \equiv h$, then for arbitrarily small $h > 0$, the first point of the continuous spectrum of the corresponding operator $-\tilde{\Delta}$ is on the left of α. Thus preceding the number α there is an infinite set of points of the spectrum of this operator. This suggests the conjecture that when the positive function $\delta(z)$ tends sufficiently slowly to zero there is an infinite set of points of the spectrum $S(-\tilde{\Delta})$ to the left of the point α.

The following theorem proves the above conjectures and makes them more precise. It may be considered to be the analogue of Kneser's conditions for the oscillatory character of the one-dimensional second-order differential operation.

Theorem 16 /32/. If for the limit-cylindrical domain Ω, defined by equation (52)

$$\lim_{z \to \infty} \sup z^2 \delta(z) < \frac{1}{8\alpha}, \tag{54}$$

where α is the first point of $C(-\tilde{\Delta})$, then to the left of the point α there are only a finite number of eigenvalues of the operator $-\tilde{\Delta}$.

If

$$\lim_{z \to \infty} \inf z^2 \delta(z) > \frac{1}{8\alpha}, \tag{55}$$

then the part of the spectrum $S(-\tilde{\Delta})$, lying to the left of the point α, consists of an infinite sequence of eigenvalues of finite multiplicity with the only limit point $\lambda = \alpha$.

Proof. Replacing $\delta(z)$ in (52) by the function $\mu(z)$, which also satisfies the condition $\mu(z) > -1$ and the limit relation (53), we obtain some limit-cylindrical domain $\tilde{\Omega}$. This domain is bounded by a part of the plane $z = 0$ and by the surface

$$\begin{aligned} x &= x(\varphi)[1 + \mu(z)] \\ y &= y(\varphi)[1 + \mu(z)] \end{aligned} \qquad (0 \leqslant \varphi < 2\pi,\ 0 \leqslant z < \infty). \tag{56}$$

Without yet indicating the method of constructing $\mu(z)$ from a given function $\delta(z)$, we transform the domain $\tilde{\Omega}$ into a cylindrical domain Z with the boundaries $z = 0$ and

$$\begin{aligned} \xi &= x(\varphi) \\ \eta &= y(\varphi) \end{aligned} \qquad (0 \leqslant \varphi < 2\pi,\ 0 \leqslant z < \infty)$$

by means of the substitution

$$x=[1+\mu(z)]\xi, \quad y=[1+\mu(z)]\eta, \quad z=z.$$

Setting

$$v(\xi, \eta, z)=u(x, y, z),$$

we obtain

$$\left.\begin{aligned}\frac{\partial u}{\partial x}&=\frac{1}{1+\mu}\frac{\partial v}{\partial \xi}, \quad \frac{\partial u}{\partial y}=\frac{1}{1+\mu}\frac{\partial v}{\partial \eta},\\ \frac{\partial u}{\partial z}&=\frac{\partial v}{\partial z}-\frac{x\mu'}{(1+\mu)^2}\frac{\partial v}{\partial \xi}-\frac{y\mu'}{(1+\mu)^2}\frac{\partial v}{\partial \eta}.\end{aligned}\right\} \tag{57}$$

Furthermore, in order to investigate the sign of the quadratic functional

$$\Phi[u]=\int\int\int_{(\Omega)}(|\nabla u|^2-\alpha|u|^2)\,dx\,dy\,dz,$$

we estimate the quantity $\left(\frac{\partial u}{\partial z}\right)^2$ from below and from above in terms of the partial derivatives of the function v with respect to the new variables.

Putting for the right-hand side of (51)

$$R=\max_{0\leqslant\varphi<2\pi} r(\varphi)$$

and introducing the notation

$$\eta(z)=\frac{R|\mu'|}{(1+\mu)^2}, \tag{58}$$

we then easily obtain from relation (57) the inequalities

$$\left(\frac{\partial u}{\partial z}\right)^2\geqslant\left(\frac{\partial v}{\partial z}\right)^2(1-2\eta)-\left[\left(\frac{\partial v}{\partial \xi}\right)^2+\left(\frac{\partial v}{\partial \eta}\right)^2\right](\eta+\eta^2),$$
$$\left(\frac{\partial u}{\partial z}\right)^2\leqslant\left(\frac{\partial v}{\partial z}\right)^2(1+2\eta)+\left[\left(\frac{\partial v}{\partial \xi}\right)^2+\left(\frac{\partial v}{\partial \eta}\right)^2\right](\eta+2\eta^2).$$

Together with (57), these inequalities lead to an estimate from above and below of the functional $\Phi[u]$ in the form

$$F_1[v]\leqslant\Phi[u]\leqslant F_2[v], \tag{59}$$

where

$$F_j[v]=\int\int_{(Z)}\int\left\{q(z)\left[\left(\frac{\partial v}{\partial \xi}\right)^2+\left(\frac{\partial v}{\partial \eta}\right)^2\right]+p_j(z)\left(\frac{\partial v}{\partial z}\right)^2-\alpha\rho(z)v^2\right\}dZ \tag{60}$$

and

$$q_j(z)=\begin{cases}1-\eta(1+\mu)^2-\eta^2(1+\mu)^2 & (j=1),\\ 1+\eta(1+\mu)^2+2\eta^2(1+\mu)^2 & (j=2),\end{cases}$$
$$p_{1,2}(z)=(1\mp2\eta)(1+\mu)^2, \quad \rho(z)=(1+\mu)^2.$$

It is obvious that for any finite function v vanishing on the boundary of the cylinder Z and twice continuously differentiable inside Z, the quadratic functional (60) may be represented in the form

$$F_j[v]=(l_j(v), v),$$

where

$$l_j[v]=-q_j(z)\left(\frac{\partial^2 v}{\partial \xi^2}+\frac{\partial^2 v}{\partial \eta^2}\right)-\frac{\partial}{\partial z}\left(p_j(z)\frac{\partial v}{\partial z}\right)-\alpha\rho(z)v. \tag{61}$$

The problem now reduces to the investigation of the negative part of the spectrum of the operator M_j defined in $\mathcal{L}_2(Z)$ by operation (61) with zero boundary condition on the boundary of the cylinder Z. Since the variables are separable in the corresponding boundary-value problem, the required investigation does not present difficulties.

Setting

$$v(\xi, \eta, z) = w(\xi, \eta)\zeta(z)$$

in the equation

$$l_j[v] = \lambda v,$$

we obtain

$$\frac{\frac{d}{dz}\left(p_j \frac{d\zeta}{dz}\right)}{q_j\zeta} = \frac{\Delta w}{w} + \frac{\lambda + \alpha\rho}{q_j},$$

whence

$$-\Delta w = kw$$

and

$$-\frac{d}{dz}\left(p_j \frac{d\zeta}{dz}\right) + (kq_j - \alpha\rho)\zeta = \lambda\zeta. \tag{62}$$

Together with the boundary condition $z(0) = 0$, the left-hand side of (62) defines some self-adjoint operator L_j in $\mathcal{L}_2(0, \infty)$. If the parameter $k = k_n$ in the left-hand side of (62) runs through the set of all the eigenvalues of the two-dimensional Laplacian $-\Delta$ in the cross section of the cylinder Z with the zero boundary condition, then the operators L_j run through some aggregate $\{L_{jn}\}_{n=1}^{\infty}$. Clearly the number of negative eigenvalues of the operator M_j is equal to the total number of negative eigenvalues of all the operators of the aggregate $\{L_{jn}\}_{n=1}^{\infty}$.

Now, assuming that relation (54) is fulfilled, we choose a number z_0 so that for some $\beta > 0$ we have for any $z > z_0$

$$\delta(z) < \frac{1-\beta}{8\alpha z^2}.$$

Furthermore, on the interval $0 \leqslant z \leqslant z_1$, where $z_1 \geqslant z_0$, we construct a positive linear function $az + b$, having the following properties: the function

$$\mu(z) = \begin{cases} az + b & (0 \leqslant z < z_1), \\ \frac{1-\beta}{8\alpha z^2} & (z_1 < z) \end{cases} \tag{63}$$

is continuous at the point z_1; the function $\eta(z)$, defined by formula (59), is less than half for all $z \geqslant 0$; the domain $\tilde{\Omega}$ contains the domain Ω. We smooth the constructed function (63) in the neighborhood of the point z_1 so that it becomes continuously differentiable at the point z_1 without ceasing to fulfill the above properties. The smoothed function is again denoted by $\mu(z)$.

With this choice of the function $\mu(z)$ in (56), the coefficient $p_1(z)$ in (62) is bounded from below by a positive number for $z \geqslant 0$, and $\lim p_1(z) = 1$ for $z \to \infty$. For sufficiently large values of k, the coefficient $q_1(z)$ in (62) is positive for all $z \geqslant 0$, and for any k and $z \to \infty$ it has the form

$$q_1(z) = k - \alpha - \frac{1-\delta}{4z^2} + O(z^{-3}). \tag{64}$$

From the properties of the coefficients of operation (61), with the exception of (64), follows the existence of a number $n=n_0$, such that for $n \geqslant n_0$, the operators L_{1n} are positive. Moreover, since for all n we have $k_n \geqslant \alpha$, it follows from (64) that for large z the coefficient $q_1(z)$ is either positive or (for $n=1$) it has the form:

$$q_1(z) = -\frac{1-\delta}{4z^2} + O(z^{-3}). \tag{65}$$

Therefore, for all $n > n_0$ and a certain N, the quadratic functional $(L_1\zeta, \zeta)$ is non-negative on all the N-finite functions $\zeta \in \mathfrak{D}_{L_{1,n}}$. This is obvious for $q_1(z) > 0$ $(z > N)$, and it follows from the non-oscillatoriness condition (39) of No. 30 for $q_1(z)$ of the form (65). Consequently, for $1 \leqslant n < n_0$, each of the operators $L_{1,n}$ may have only a finite number of negative eigenvalues.

It has thus been established that the operator M_1 has only a finite number of negative eigenvalues. But then, by virtue of the first part of the double inequality (59), the operator $-\tilde{\Delta}$ in the limit-cylindrical domain $\tilde{\Omega}$ has only a finite number of negative eigenvalues to the left of the point α. Finally, since $\Omega \supset \tilde{\Omega}$, then according to Lemma 2 the operator $-\tilde{\Delta}$ in $\mathscr{L}_2(\Omega)$ has only a finite number of eigenvalues smaller than α. Thus, the first part of the theorem is proved.

Turning to the demonstration of the second part of the theorem, we assume that relation (65) is fulfilled and choose a number z_0 such that for some $\beta > 0$ we have for any $z \geqslant z_0$

$$\delta(z) > \frac{1+2\beta}{8\alpha z^2}.$$

Setting in (56)

$$\mu(z) = \frac{1+\beta}{8\alpha z^2}$$

for $z \geqslant z_0$, then $\lim p_2(z) = 1$ for $z \to \infty$ and

$$q_2(z) = k - \alpha - \frac{1+\beta}{4z^2} + O(z^{-3}).$$

It is then possible to choose a number $z_1 \geqslant z_0$ such that for $z \geqslant z_1$

$$p_1(z) < 1+\beta. \tag{66}$$

Let $-\tilde{\Delta}_1$ denote the operator defined by the Laplacian $-\Delta$ in the limit-cylindrical domain $\tilde{\Omega}_1$, bounded by the surfaces $z = z_1$ and (56) with zero boundary condition. The corresponding operator M_2 operates in $\mathscr{L}_2(Z_1)$, where Z_1 is that part of the cylinder Z lying above the plane $z = z_1$. The operators $L_{2,n}$ operate in $\mathscr{L}_2(z_1, \infty)$. We then have for $n = 1$

$$q_2(z) = -\frac{1+\beta}{4z^2} + O(z^{-3}). \tag{67}$$

On the basis of the oscillatoriness condition (40) of No. 30 we conclude from (66) and (67) that the set of negative eigenvalues of the operator $L_{2,n}$ is infinite for $n = 1$.

It has thus been established that the operator M_2 has an infinite set of negative eigenvalues. But then, by virtue of the second part of the double inequality (59), the operator $-\tilde{\Delta}_1$ in the limit-cylindrical domain $\tilde{\Omega}_1$ has an infinite set of eigenvalues to the left of the point α. Finally, since $\tilde{\Omega}_1 \subset \Omega_1$, then according to Lemma 2 the operator $-\tilde{\Delta}$ in $\mathscr{L}_2(\Omega)$ has an infinite sequence of eigenvalues to the left of the point α. The theorem is completely proved.

It follows from Theorem 16 that under conditions (54) or (55) the number of eigenvalues of the operator $-\tilde{\Delta}$ lying to the left of the lower bound

of its continuous spectrum is changed by only a finite number when the domain Ω undergoes a finite deformation.

Let $m(\varepsilon)$ denote the number of eigenvalues of the operator $-\tilde{\Delta}$ that are smaller than $\alpha-\varepsilon$. If the perturbation $\delta(z)$ satisfies condition (55), it is obvious that $m(\varepsilon)$ increases the faster when $\varepsilon \downarrow 0$, the slower the function $\delta(z)$ decreases when $z \to \infty$. To obtain the corresponding estimates we assume that the function $\delta(z)$ is positive and monotonically decreasing. For small $\varepsilon > 0$, let $a(\varepsilon)$ denote a number having the property that the first eigenvalue of the boundary-value problem for the two-dimensional Laplacian $-\Delta$ in the section $z = a(\varepsilon)$ with zero boundary condition on its boundary is equal to $\alpha-\varepsilon$, and let ω_ε denote the finite section of the domain Ω, cut from it by the plane $z = a(\varepsilon)$. Furthermore, let $k(\varepsilon)$ denote the number of eigenvalues of the operator $-\Delta$ in $\mathcal{L}_2(\omega_\varepsilon)$ with zero boundary condition on the boundary of ω_ε which are smaller than $\alpha-\varepsilon$. By $k'(\varepsilon)$ we donote the number of eigenvalues, smaller than $\alpha-\varepsilon$, in the boundary-value problem arising from the former by replacing the zero boundary condition on the boundary $z = a(\varepsilon)$ by the condition $\frac{\partial u}{\partial n} = 0$.

Since the operator defined by the operation $-\Delta$ in $\mathcal{L}_2(\Omega - \omega_\varepsilon)$ with the condition $\frac{\partial u}{\partial n} = 0$ on the section $z = a(\varepsilon)$ and the zero condition on the remaining part of the boundary of $\Omega - \omega_\varepsilon$ obviously does not have points of the spectrum to the left of $\alpha-\varepsilon$, then in accordance with Lemma 1

$$k(\varepsilon) < m(\varepsilon) < k'(\varepsilon). \tag{68}$$

Dividing the interval $0 \leqslant z \leqslant a(\varepsilon)$ into parts by the points $z_0 = 0, z_1, z_2, \ldots, z_{n-1}, z_n = a(\varepsilon)$, we construct from the cylinders of heights $z_{l+1} - z_l$ $(l = 0, 1, \ldots, n-1)$, respectively, inscribed in and circumscribed about the domain ω_ε, the domains λ_ε and λ'_ε. Moreover, we consider the two boundary-value problems defined by the operation $-\Delta$ in the domains λ_ε and λ'_ε. In this case we give zero conditions on the cylindrical sections of the boundaries of these domains, and on the $z = z_l$ $(l = 0, 1, \ldots, n)$ the condition $u = 0$ in the first case, and $\frac{\partial u}{\partial n} = 0$ in the second case. Let $\nu(\varepsilon)$ and $\nu'(\varepsilon)$ denote the number of eigenvalues of these boundary-value problems which are smaller than $\alpha-\varepsilon$. It follows from the variational principles of R. Courant that these numbers satisfy $\nu(\varepsilon) < k(\varepsilon)$ and $k'(\varepsilon) < \nu'(\varepsilon)$. These relations together with (68) give

$$\nu(\varepsilon) < m(\varepsilon) < \nu'(\varepsilon). \tag{69}$$

For each of the cylinders constituting the domain $\lambda(\varepsilon)$, the determination of the number of eigenvalues smaller than $\alpha-\varepsilon$ reduces to solving the inequalities

$$\frac{\alpha}{[1+\delta(z_l)]^2} + \frac{m_l^2\pi^2}{(z_l - z_{l-1})^2} < \alpha - \varepsilon \qquad (l = 1, 2, \ldots, n)$$

with respect to the natural numbers m_l. The determination of the number of eigenvalues, smaller than $\alpha-\varepsilon$, for the cylinders which constitute the domain λ'_ε, reduces to similar inequalities.

Thus, we easily arrive at the inequalities

and

$$\nu(\varepsilon) > \frac{1}{\pi}\sum_{l=1}^{n}(z_l - z_{l-1})\sqrt{\alpha-\varepsilon-\frac{\alpha}{(1+\delta(z_l))^2}} + (n-1)$$

$$\nu'(\varepsilon) < \frac{1}{\pi}\sum_{l=1}^{n}(z_l - z_{l-1})\sqrt{\alpha-\varepsilon-\frac{\alpha}{(1+\delta(z_{l-1}))^2}} + 2n,$$

which, together with (69), give for $m(\varepsilon)$ the bounds

$$-2n+\frac{1}{\pi}\check{S}_n(\varepsilon)<m(\varepsilon)<\frac{1}{\pi}\hat{S}_n(\varepsilon)+2n, \tag{70}$$

where $\check{S}_n$ and $\hat{S}_n$ are the lower and upper Darboux sums for the integral

$$J(\varepsilon)=\int_0^{a(\varepsilon)}\sqrt{\alpha-\varepsilon-\frac{\alpha}{(1+\delta(z))^2}}\,dz.$$

With certain restrictions on the perturbation $\delta(z)$, it is possible to obtain from (70) the asymptotic formula

$$m(\varepsilon)=\frac{1+o(1)}{\pi}\int_0^{a(\varepsilon)}\sqrt{\alpha-\varepsilon-\frac{\alpha}{[1+\delta(z)]^2}}\,dz. \tag{71}$$

for $\varepsilon\downarrow 0$ /32/.

The representation of the class of perturbations $\delta(z)$ for which formula (71), given in /32/, is valid, is somewhat cumbersome and is therefore omitted. However, in contrast to the method given in the paper /84/, where the related one-dimensional problem was considered, the above derivation of formula (71) enables the one-dimensional asymptotic formula of N. Rosenfeld to be generalized to the two-term one-dimensional operations of any order considered in Chapter II. The generalization of the asymptotic formula of N. Rosenfeld to the multi-dimensional operation (1) in the whole space $\mathfrak{E}$, assuming five-fold differentiability of the potential as well as other restrictive conditions, is given in /18/.

Deviating from purely analytic considerations, we confine ourselves to the proof of the following theorem, which follows from /32/.

Theorem 17. If the perturbation $\delta(z)$ of the cylindrical domain (51) satisfies

$$0<\liminf_{z\to\infty} z^{\varkappa}\delta(z)\leqslant\limsup_{z\to\infty} z^{\varkappa}\delta(z)<\infty,$$

for some positive $\varkappa<2$, then there exist constants $A>0$ and $B>0$, such that for any $\varepsilon>0$

$$A\left(\frac{1}{\varepsilon}\right)^{\frac{2-\varkappa}{2\varkappa}}<m(\varepsilon)<B\left(\frac{1}{\varepsilon}\right)^{\frac{2-\varkappa}{2\varkappa}} \tag{72}$$

Proof. By virtue of Lemma 2, it is sufficient to demonstrate the theorem for the perturbation $\delta(z)=C(1+z)^{-\varkappa}$.

Taking some positive number $\beta<1$ and a natural number n, we define ε by the equality

$$\alpha-\varepsilon=\frac{\alpha}{(1+C\beta^n)^2}.$$

The number β is kept fixed during the whole proof. It is obvious that for $\varepsilon\downarrow 0$

$$n=O(\ln\varepsilon). \tag{73}$$

In making the sums $\check{S}_n(\varepsilon)=\check{S}(\varepsilon)$ and $\hat{S}_n(\varepsilon)=\hat{S}(\varepsilon)$ we assume

$$\delta(z_l)=C\beta^l \qquad (l=0,\ 1,\ \ldots,\ n),$$

so that

$$z_l=\beta^{-\frac{l}{\varkappa}}-1$$

and

$$z_l-z_{l-1}=\left(1-\beta^{\frac{1}{\varkappa}}\right)\beta^{-\frac{l}{\varkappa}}.$$

Now, for the lower Darboux sum we have

$$\check{S}(\varepsilon)=\sum_{l=1}^{n}(z_l-z_{l-1})\sqrt{\alpha-\varepsilon-\frac{\alpha}{(1+\delta(z_l))^2}}=\sqrt{\alpha}\left(1-\beta^{\frac{1}{\varkappa}}\right)\sum_{l=1}^{n}\beta^{-\frac{l}{\varkappa}}\sqrt{\frac{1}{(1+C\beta^n)^2}-\frac{1}{(1+C\beta^l)^2}}>$$

$$>\sqrt{\alpha}\left(1-\beta^{\frac{1}{\varkappa}}\right)\sum_{l=1}^{n}\beta^{-\frac{l}{\varkappa}}\sqrt{\frac{1}{(1+C\beta^n)^2}-\frac{1}{(1+C\beta^{l-1})^2}},$$

whence for sufficiently small $\varepsilon>0$ we obtain the estimate

$$\check{S}(\varepsilon)>\gamma_1\left(\frac{1}{\varepsilon}\right)^{\frac{2-\varkappa}{2\varkappa}}. \tag{74}$$

where the constant γ_1 is independent of ε.

For the upper Darboux sum we have

$$\hat{S}(\varepsilon)=\sum_{l=1}^{n}(z_l-z_{l-1})\sqrt{\alpha-\varepsilon-\frac{\alpha}{(1+\delta(z_{l-1}))^2}}=$$

$$=\left(1-\beta^{\frac{1}{\varkappa}}\right)\sum_{l=1}^{n}\beta^{-\frac{l}{\varkappa}}\sqrt{\frac{\alpha}{(1+C\beta^n)^2}-\frac{\alpha}{(1+C\beta^{l-1})^2}}=$$

$$=\sqrt{\alpha}\left(1-\beta^{\frac{1}{\varkappa}}\right)\sum_{l=1}^{n}\beta^{-\frac{l}{\varkappa}}\frac{\sqrt{(1+C\beta^{l-1})^2-(1+C\beta^n)^2}}{(1+C\beta^n)(1+C\beta^l)}<$$

$$<\sqrt{\alpha}\left(1-\beta^{\frac{1}{\varkappa}}\right)\sum_{l=1}^{n}\beta^{-\frac{l}{\varkappa}}\frac{\sqrt{2C\beta^{l-1}+C^2\beta^{2(l-1)}}}{(1+C\beta^n)^2},$$

whence for sufficiently small $\varepsilon>0$ we obtain the estimate

$$\hat{S}(\varepsilon)<\gamma_2\left(\frac{1}{\varepsilon}\right)^{\frac{2-\varkappa}{2\varkappa}} \tag{75}$$

with the constant γ_2 independent of ε.

Comparing (70), (73), (74) and (75), we obtain the required inequality (72)

Replacing the factor $1+\delta(z)$ in (52) by $\delta(z)$, the domain Ω contracts without limit when $z\to\infty$. The spectrum of the operator $-\tilde{\Delta}$ in this domain is, according to Theorem 4, discrete. If the volume V of such a limit-cylindrical domain, with zero radius at infinity, is finite, then, assuming that the perturbation $\delta(z)$ is monotonically decreasing, the ordinary asymptotic formula

$$m(\lambda)=\frac{1+o(1)}{6\pi^2}\lambda^{3/2}V,$$

for $\lambda\to\infty$ remains valid for the number $m(\lambda)$ of eigenvalues smaller than λ; for $V=\infty$ we have

$$m(\lambda)=\frac{1+o(1)}{6\pi^2}\lambda^{3/2}\int\limits_{\delta>\frac{\alpha}{\sqrt{\lambda}}}\delta^2(z)\,dz,$$

where α is the first eigenvalue of the boundary-value problem induced by the operator $-\tilde{\Delta}$ in the part of the plane $z=z_0$ bounded by the curve $r=r(\varphi)$ /32/.

Chapter V

THE CHARACTER OF THE SPECTRUM OF THE SCHROEDINGER DIFFERENTIAL OPERATOR AND THE BEHAVIOR OF THE SOLUTIONS OF THE CORRESPONDING HOMOGENEOUS EQUATION

51. Formulation of the problem. Let L be the minimal operator, defined in the whole space $\mathfrak{E}_m$ by the operation

$$-\Delta u + q(P)u \tag{1}$$

with the potential $q(P)$ bounded from below, such that $L = L^*$. The subject of the present chapter is the investigation of the connection between the behavior, at infinity, of the solutions $\varphi(P; \lambda)$ of the equation

$$-\Delta u + q(P)u = \lambda u \tag{2}$$

and the position of the point λ with respect to the spectrum of the operator L.

The condition of semi-boundedness from below of the potential is assumed to be fulfilled throughout the whole chapter. In this connection, q_0 denotes the quantity satisfying the inequality

$$-\infty < q_0 \leqslant \inf_{P \in \mathfrak{E}_m} q(P).$$

For a preliminary orientation in the problem, we consider the simplest self-adjoint one-dimensional differential operator L_1 defined on the semi-axis $x \geqslant 0$ by the operation $-\frac{d^2}{dx^2}$ with the boundary condition $y'(0) = -\theta y(0)$. The continuous part $C(L_1)$ of the spectrum of this operator coincides with the semi-axis $\lambda \geqslant 0$, and the discrete part $D(L_1)$ reduces to the single point $\lambda = -\theta^2$. The only solution up to a multiplicative constant

$$\varphi(x; \lambda) = (i\sqrt{\lambda} - \theta)e^{i\sqrt{\lambda}x} + (i\sqrt{\lambda} + \theta)e^{-i\sqrt{\lambda}x}$$

of the equation

$$y'' + \lambda y = 0, \tag{3}$$

satisfying the given boundary condition, possesses the following properties for $x \to \infty$:

1) Excluding the possibility that the point λ belongs to an isolated part of the set $D(L_1)$, the function $\varphi(x; \lambda)$ increases more slowly, the nearer the point λ to the set $C(L_1)$.

2) If λ is an isolated point of the set $D(L_1)$, then the function $\varphi(x; \lambda)$ decreases exponentially, and the rate of this decrease is larger, the further the point λ from the set $C(L_1)$.

3) If $\lambda \in C(L_1)$, then the function $\varphi(x; \lambda)$ is bounded for $x \to \infty$.

It will be shown in the present chapter that the properties 1) — 3) of the solution of equation (3) reflect to some extent the situation existing in the general case of equation (2), provided we assume that the potential is bounded from below.

The following results of Nos. 52-55 are due to E. E. Shnol' /108/.

52. Some a priori estimates for the solutions of the homogeneous elliptic equation. We now establish two lemmas /108 (4)/ which are important for all that follows; they deal with the solutions of the equation

$$-\Delta u + q(P)\,u = 0 \tag{4}$$

with the potential $q(P) \geqslant q_0 > -\infty$. Roughly, the meaning of these lemmas is that for a fixed integral

$$\int\limits_{|PQ|=R} |u(Q)|^2\, d\omega_Q$$

neither the magnitude of the gradient of the solution in the sphere $|PQ| \leqslant R$, nor the value $u(P)$ of the solution in the center of this sphere can be too large.

Lemma 1. Let R and $R_1 (R_1 < R)$ be given, and let the point $P \in \mathfrak{E}_m$ be chosen arbitrarily. For any solution of the equation (4) we have the inequality

$$\int\limits_{|PQ|<R_1} |\nabla u(Q)|^2\, d\omega_Q < C \int\limits_{|PQ|<R} |u(Q)|^2\, d\omega_Q. \tag{5}$$

where the constant C depends solely on q_0.

Lemma 2. Let $R > 0$ be given, and let the point $P \in \mathfrak{E}_m$ be chosen arbitrarily. For any solution of equation (4) we have the inequality

$$|u(P)|^2 < K \int\limits_{|PQ|<R} |u(Q)|^2\, d\omega_Q. \tag{6}$$

where the constant K depends solely on q_0.

In the proof of both theorems and further on we assume for definiteness that $m = 2$.

Let $\theta(Q)$ be a smooth function in $\mathfrak{E}_2$, satisfying the conditions

$$\theta(Q) = \begin{cases} 1 & \text{for } |PQ| \leqslant R_1, \\ 0 & \text{for } |PQ| \geqslant R, \end{cases}$$
$$0 \leqslant \theta(P) \leqslant 1 \qquad (R_1 \leqslant |PQ| \leqslant R).$$

Applying the first Green formula

$$\int\limits_{|PQ|<R} l[u]\, v\, d\omega_Q = \int\limits_{|PQ|<R} \left\{ \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} + quv \right\} d\omega_Q - \int\limits_{|PQ|=R} \frac{\partial u}{\partial n}\, v\, d\sigma_Q$$

to the solution $u(Q)$ and to the function $v(Q) = \theta(Q)\,u(Q)$, we obtain

$$0 = \int\limits_{|PQ|<R} \theta \left\{ \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 \right\} d\omega_Q + \int\limits_{|PQ|<R} u \left\{ \frac{\partial \theta}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial \theta}{\partial y}\frac{\partial u}{\partial y} \right\} d\omega_Q + \int\limits_{|PQ|<R} q\theta u^2\, d\omega_Q,$$

but

$$\int\limits_{|PQ|<R}\left\{\frac{\partial\theta}{\partial x}\frac{\partial u^2}{\partial x}+\frac{\partial\theta}{\partial y}\frac{\partial u^2}{\partial y}\right\}d\omega_Q=-\int\limits_{|PQ|<R}u^2\Delta\theta\,d\omega_Q,$$

so that

$$\int\limits_{PQ|<R}\theta\left\{\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right\}d\omega_Q=\frac{1}{2}\int\limits_{|PQ|<R}u^2\Delta\theta\,d\omega_Q-\int\limits_{|PQ|<R}q\theta u^2\,d\omega_Q,$$

whence

$$\int\limits_{|PQ|<R_1}\left\{\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right\}d\omega_Q<\frac{1}{2}\int\limits_{|PQ|<R}u^2\Delta\theta\,d\omega_Q-q_0\int\limits_{|PQ|<R}\theta u^2 d\omega_Q \qquad (7)$$

and consequently

$$\int\limits_{|PQ|<R_1}|\nabla u(Q)|^2 d\omega_Q<C\int\limits_{|PQ|<R}|u(Q)|^2\,d\omega_Q. \qquad (8)$$

Lemma 1 is thus proved. Turning to the proof of Lemma 2, we first choose the radius R of the circle $|P_0Q|\leqslant R$ so small, that the first eigenvalue of the internal boundary-value problem for the Laplace equation with the condition $u(Q)=0$ for $|P_0Q|=R$ is larger than $|q_0|$. With this choice of the number R, all the eigenvalues of such a boundary-value problem for equation (4) are positive. From the variational principle of R. Courant it follows that all the eigenvalues of the internal boundary-value problem for equation (4) with the condition $u(Q)|_{Q\in\Gamma}=0$ are also positive if only Γ is the boundary of a domain lying within the circle $|P_0Q|<R$. Hence it follows in particular, that any solution of equation (4) that vanishes on the boundary of some domain lying within a circle of radius R is identically zero in this domain.

Using this, we show that the Green functions $G(P, Q)$ and $G_0(P, Q)$ of the boundary-value problems for equations (4) and

$$-\Delta u+q_0u=0$$

with the condition $u(Q)=0$ for $|P_0Q|=R$ respectively, satisfy the inequality

$$G(P, Q)\leqslant G_0(P, Q) \qquad (9)$$

for $|P_0P|<R$, $|P_0Q|<R$. By virtue of the relation

$$K_0(P, Q)=K(P, Q)+\int\limits_{|P_0M|<R}K(P, M)[q(M)-q_0]K_0(M, Q)\,d\omega_M$$

it is sufficient to show that the latter integrand is non-negative. For this, it is sufficient to prove that the Green function $G(P,Q)$ is non-negative for $|P_0P|<R$, $|PQ|<R$.

Assuming the opposite, we obtain $G(P_1, Q_1)<0$, and since for points Q close to P_1 it is obvious that $G(P_1, Q)>0$, there exists inside the circle $|P_0Q|\leqslant R$ a domain on whose boundary $G(P_1, Q)=0$; but then in this domain $G(P_1, Q)\equiv 0$, which is absurd.

Inequality (9) is thus established and we may proceed directly to the derivation of inequality (6).

We choose numbers R_1 and $R_2 (0 < R_1 < R_2 < R)$ and construct a function smooth in $\mathfrak{E}_2$, which satisfies the conditions $\theta(Q)$

$$\theta(Q) = \begin{cases} 1 & \text{for } |P_0Q| \leqslant R_1, \\ 0 & \text{for } |P_0Q| \geqslant R_2, \end{cases}$$
$$0 \leqslant \theta(Q) \leqslant 1 \qquad (R_1 < |P_0Q| < R_2).$$

For any solution $u(Q)$ of equation (5) we have

$$\|l[\theta u]\| = \left\| \frac{\partial \theta}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial \theta}{\partial y}\frac{\partial u}{\partial y} + u\,\Delta\theta \right\| \leqslant C_1 \left\{ \int\limits_{|P_0Q| < R_2} |\nabla u|^2 \, d\omega_Q \right\}^{\frac{1}{2}} + C_2 \left\{ \int\limits_{|P_0Q| < R_2} |u|^2 \, d\omega_Q \right\}^{\frac{1}{2}},$$

and according to Lemma 1,

$$\|l[\theta u]\|^2 \leqslant C \int\limits_{|P_0Q| < R} |u|^2 \, d\omega_Q. \tag{10}$$

Using inequalities (9), (10) and the Cauchy-Bunyakovskii inequality, we obtain for the quantity

$$u(P_0) = \theta(P_0)\, u(P_0) = \int\limits_{|P_0Q| < R} G(P_0, Q)\, l[\theta(Q)\, u(Q)]\, d\omega_Q \tag{11}$$

the estimate

$$|u(P_0)|^2 \leqslant \int\limits_{|P_0Q| < R} G^2(P_0, Q)\, Q\omega_Q \cdot \int\limits_{|P_0Q| < R} |l[\theta u]|^2 \, d\omega_Q \leqslant$$
$$\leqslant \int\limits_{|P_0Q| < R} G_0^2(P_0, Q)\, d\omega_Q \cdot \|l[\theta u]\|^2 \leqslant K \int\limits_{|P_0Q| < R} |u|^2 \, d\omega_Q.$$

This estimate is established for sufficiently small R, but since the right-hand side of the inequality increases with R, Lemma 2 is completely proved.

Although the above proof is invalidated for $m > 3$ by the divergence of the integral

$$\int\limits_{|P_0Q| < R} G^2(P_0, Q)\, d\omega_Q,$$

this can easily be corrected, since in (11) the actual integration is performed over the annular domain $R_1 \leqslant |P_0Q| \leqslant R_2$.

These lemmas are close to a geometrical theorem of S. N. Bernshtein connected with elliptic differential equation (/12/, see also /2/).

53. Behavior of the solutions at points of the discrete part of the spectrum. If $\lambda \in D(L)$ and $\varphi(P; \lambda)$ is the corresponding eigenfunction of the operator L, then $\varphi(P; \lambda) \in \mathscr{L}_2(\mathfrak{E}_m)$. On the other hand, by virtue of Lemma 2,

$$|\varphi(P; \lambda)| \leqslant K \int\limits_{|PQ| < R} |\varphi(Q; \lambda)|^2 \, d\omega_Q.$$

Passing in this inequality to the limit for $|OP| \to \infty$, we obtain

$$\lim_{|OP| \to \infty} |\varphi(P; \lambda)| = 0.$$

Thus, we have the following theorem.

Theorem 1 /108 (4)/. Every eigenfunction $\varphi(P;\lambda)$ *of the operator* L *tends to zero for* $|OP|\to\infty$.

If λ is an isolated point of the set $D(L)$, then as is shown by the following Theorem, it is possible to determine a "lower bound" for the rate at which the function $\varphi(P;\lambda)$ tends to zero.

Theorem 2 /108 (4)/. If the distance of the eigenvalue λ *of the operator* L *from the set* $C(L)$ *is equal to* d, *then we have for all* $P\in\mathfrak{E}_m$

$$|\varphi(P;\lambda)|\leqslant Ce^{-\mu r}\qquad (r=|OP|), \tag{12}$$

where $\mu=\frac{1}{2}\ln\left(1+\frac{\eta^2}{a^2}\right)$, η *being any number smaller than* d, *and the constant* a *depends only on* q_0.

Proof. Let the distance of the eigenvalue λ of the operator L from $C(L)$ equal d and let $\varphi(P;\lambda)$ be an eigenfunction of the operator L. Since $\varphi(P;\lambda)\in\mathfrak{L}_2(\mathfrak{E}_m)$, then the function

$$J(r)=\int\limits_{|OP|>r}|\varphi(P,\lambda)|^2d\omega_P$$

tends to zero if $r\to\infty$, and when $\mu=0$, there exists a constant $C(\mu)$ such that for all $r\geqslant 0$

$$J(r)\leqslant C(\mu)e^{-\mu r}. \tag{13}$$

Let μ_0 denote the exact upper bound of the set of values of μ for which there exists a constant $C(\mu)$ such that for all $r\geqslant 0$ inequality (13) is satisfied. Obviously $\mu_0\geqslant 0$, and if $\mu_0\neq\infty$, then for each $\delta>0$ and each C there exists a sequence $r_n\to\infty$ satisfying

$$J(r_n)>Ce^{-(\mu_0+\delta)r_n}. \tag{14}$$

On the other hand, for each $\delta>0$ there is a number C, dependent on δ $(C=C_\delta=C(\mu_0-\delta))$, such that we have for all $r\geqslant 0$ the inequality

$$J(r)<Ce^{-(\mu_0-\delta)r}. \tag{15}$$

Fixing the number $\delta>0$, we construct a sequence $\rho_n\to\infty$, for which

$$J(\rho_n)\leqslant e^{\mu_0+\delta}J(\rho_n+1). \tag{16}$$

Such a sequence exists, since if not we would have $J(r)<e^{-(\mu_0+\delta)}J(r-1)$ for any $r>r_1$, whence follows the inequality $J(r)<Ce^{-(\mu_0+\delta)r}$ for some C. This is incompatible with (14).

We now construct the system of functions $\theta_n(P)$ according to the formula

$$\theta_n(P)=\theta_n(r)=\theta(r-\rho_n),$$

where $\theta(r)$ is a smooth function for $r>0$, satisfying the conditions

$$\theta(r)=\begin{cases}0 & \text{for } r\leqslant\frac{1}{3},\\ 1 & \text{for } r\geqslant\frac{2}{3},\end{cases} \tag{17}$$

and set

$$0\leqslant\theta(r)\leqslant 1\qquad\left(\frac{1}{3}<r<\frac{2}{3}\right)$$

$$\varphi_n(P)=\theta_n(P)\varphi(P;\lambda).$$

For the norm of the function $\varphi_n(P)\in\mathfrak{D}_L$ we obtain

$$\|\varphi_n\|^2\geqslant J(\rho_n+1)\geqslant e^{-(\mu_0+\delta)}J(\rho_n), \tag{18}$$

and for the norm of the function $l[\varphi_n]-\lambda\varphi_n$, we have, as in 52,

$$\|l[\varphi_n]-\lambda\varphi_n\| \leqslant C_1\left\{\int\limits_{\rho_n+\frac{1}{3}<|OP|<\rho_n+\frac{2}{3}} |\nabla\varphi|^2\,d\omega_P\right\}^{1/2} + C_2\left\{\int\limits_{\rho_n<|OP|<\rho_n+1} |\varphi|^2\,d\omega_P\right\}^{1/2}.$$

Using Lemma 1 and inequalities (16) and (18), we obtain

$$\|l[\varphi_n]-\lambda\varphi_n\|^2 < C\int\limits_{\rho_n<|OP|<\rho_n+1} |\varphi(P;\lambda)|^2\,d\omega_P =$$

$$= C\{J(\rho_n)-J(\rho_n+1)\} \leqslant C\{J(\rho_n)-e^{-(\mu_0+\delta)}J(\rho_n)\} =$$

$$= CJ(\rho_n)\{1-e^{-(\mu_0+\delta)}\} \leqslant Ce^{(\mu_0+\delta)}\{1-e^{-(\mu_0+\delta)}\}\|\varphi_n\|^2,$$

so that

$$\|L\varphi_n-\lambda\varphi_n\| < a\left(e^{\mu_0+\delta}-1\right)^{1/2}\|\varphi_n\|.$$

From this inequality on the basis of Theorem 10 of No. 3 we conclude that $d < a\sqrt{e^{\mu_0+\delta}-1}$, and since the quantity $\delta>0$ is arbitrary, $d \leqslant a\sqrt{e^{\mu_0}-1}$, so that for $\eta<d$ we have

$$\mu_0 \geqslant \ln\left(1+\frac{\eta^2}{a^2}\right).$$

Thus, by virtue of (15), for any $\mu<\mu_0$ and particularly for $\mu=\ln\left(1+\frac{\eta^2}{a^2}\right)$, where η is any number smaller than d, there exists a constant $C=C(\eta)$, such that for all $r\geqslant 0$

$$\int\limits_{|OP|>r} |\varphi(P;\lambda)|^2\,d\omega_P \leqslant Ce^{-\mu r}. \tag{19}$$

From this integral estimate one easily obtains, using Lemma 2, the estimate (12) for $\varphi(P;\lambda)$.

In fact, if for some $\eta_1<d$ and $C_1>0$ a sequence $r_n\to\infty$ was found, for which

$$|\varphi(P_n;\lambda)| > C_1e^{-\mu_1 r_n},$$

where $|OP_n|=r_n$ and $\mu_1=\frac{1}{2}\ln\left(1+\frac{\eta_1^2}{a^2}\right)$, then by virtue of Lemma 2 we would have

$$\int\limits_{|P_nP|<1} |\varphi(P;\lambda)|^2\,d\omega_P > C_2e^{-2\mu_1 r_n}, \tag{20}$$

where $C_2>0$.

For $\eta<\eta_1<d$, inequality (20) contradicts the established relation (18).

Thus, for the case $\mu_0\neq\infty$ the theorem is proved. If $\mu_0=\infty$ then for any μ and the corresponding value $C=C(\mu)$ the estimate (19) is valid, and it follows from Lemma 2 that up to a multiplicative constant such an estimate is valid for $\varphi(P;\lambda)$, so that the theorem is completely proved.

If the spectrum of the operator L is discrete, which, for example, is the case under the condition

$$\lim_{|OP|\to\infty} q(P)=\infty,$$

then, by virtue of the above theorem, each eigenfunction of the operator L decreases faster than $e^{-\mu r}$ for any $\mu > 0$. Thus, for example, the eigenfunctions of the one-dimensional harmonic oscillator

$$l[y] = -\frac{d^2y}{dx^2} + x^2y \qquad (-\infty < x < \infty)$$

decrease as the product of a polynomial by $e^{-(\frac{x^2}{2})}$/61 (1)/.

Theorem 2 of the present subsection reinforces the corresponding results of Wintner /23 (4)/ and Putnam /79 (3)/ on the one-dimensional differential operators.

54. An estimate of the distance of a point from the spectrum by the behavior of the solution at this point. The following theorem establishes that if for a given λ the solution $\varphi(P; \lambda)$ increases not very rapidly for $|OP| \to \infty$, then the distance of the point λ from the spectrum of the operator L is not very large.

Theorem 3 /108 (4)/. If for a given λ and for some solution $\varphi(P; \lambda)$ the following inequality is satisfied for all $P \in \mathfrak{E}_m$

$$|\varphi(P; \lambda)| < Ce^{\mu r}, \tag{21}$$

then the distance of the point λ from $S(L)$ does not exceed $d = a\sqrt{e^{2\mu} - 1}$, where $a = a(q_0)$.

Proof. The function $J(r)$, defined by the equality

$$J(r) = \int\limits_{|OP| < r} |\varphi(P, \lambda)|^2 d\omega_P,$$

satisfies, by virtue of (21), the condition

$$J(r) < e^{(2\mu+\delta)r} \tag{22}$$

for any $\delta > r_0$ and for any $r > r_0$, where $r_0 = r(\delta)$.

Fixing the number $\delta_1 > 0$, we construct a sequence $\rho_n \to \infty$, for which

$$J(\rho_n + 1) < e^{2\mu+\delta_1} J(\rho_n). \tag{23}$$

Such a sequence exists, since if not we would have for any $r > r_1$

$$J(r) > e^{2\mu+\delta_1} J(r-1),$$

whence for a certain $C > 0$ it follows that

$$J(r) > Ce^{(2\mu+\delta_1)r},$$

which is incompatible with (21) for $\delta < \delta_1$.

We now construct the system of functions $\theta_n(P)$ by the formula

$$\theta_n(P) = \theta_n(r) = \theta(r - \rho_n),$$

where $\theta(r)$ is a smooth function for $r > 0$ and satisfies conditions (17). We set

$$\varphi_n(P) = [1 - \theta_n(P)]\,\varphi(P; \lambda). \tag{24}$$

For the norm of the function $\varphi_n(P) \in \mathfrak{D}_L$ we have

$$\|\varphi_n\|^2 > J(\rho_n), \tag{25}$$

and for the norm of the function $l[\varphi_n] - \lambda\varphi_n$, as in the previous subsection,

we obtain

$$\|l[\varphi_n]-\lambda\varphi_n\| \leqslant C_1\left\{\int\limits_{\rho_n+\frac{1}{3}\leqslant|OP|\leqslant\rho_n+\frac{2}{3}} |\nabla\varphi|^2\,d\omega_P\right\}^{1/2}+$$
$$+C_2\left\{\int\limits_{\rho_n+\frac{1}{3}\leqslant|OP|\leqslant\rho_n+\frac{2}{3}} |\varphi|^2\,d\omega_P\right\}^{1/2} \qquad (26)$$

Using Lemma 1 and inequalities (23), (25), we obtain

$$\|l[\varphi_n]-\lambda\varphi_n\|^2 < C\int\limits_{\rho_n<|OP|\leqslant\rho_n+1} |\varphi(P;\lambda)|^2\,d\omega_P = \qquad (27)$$
$$= C[J(\rho_n+1)-J(\rho_n)] < C(e^{2\mu+\delta_1}-1)J(\rho_n) <$$
$$< C(e^{2\mu+\delta_1}-1)\|\varphi_n\|^2,$$

so that

$$\|L\varphi_n-\lambda\varphi_n\| < a\sqrt{e^{2\mu+\delta_1}-1}\,\|\varphi_n\|. \qquad (28)$$

From this inequality it follows by virtue of the spectral expansion that the distance of the point λ from the spectrum $S(L)$ does not exceed $a\sqrt{e^{2\mu+\delta_1}-1}$, and since the number $\delta_1>0$ is arbitrary, this distance does not exceed $a\sqrt{e^{2\mu}-1}$, Q. E. D.

From the above theorem directly follows

Theorem 4 /108 (4)/. If some solution $\varphi(P;\lambda)$ satisfies the inequality

$$|\varphi(P;\lambda)| < C(\varepsilon)e^{\varepsilon r},$$

for any $\varepsilon>0$ and for all $P\in\mathfrak{G}_m$, then $\lambda\in S(L)$.

In particular, for $\varepsilon=0$ follows

Theorem 5 /108 (1)/. If for a given λ there exists a bounded solution $\varphi(P;\lambda)$ then $\lambda\in S(L)$.

The converse statement will be considered in Nos 55 and 58.

The result established by Theorem 3 is far from being the best possible. This is observed, for example, when applying Theorem 3 to the one-dimensional Schroedinger operator on the semi-axis $x\geqslant 0$ with the bounded potential $|q(x)|<M$. In fact, in this case for any solution of equation (2) we obviously have (see, for example, /67 (1)/)

$$|\varphi(x;\lambda)| < A\exp M\lambda^{-1/2}x.$$

On the other hand, when passing from (26) to (27) use is made of the estimate (8) to which the dependence of the constant C on λ is linear, since q_0 is to be replaced in inequality (7) by $q_0-\lambda$ when replacing equation (4) by equation (2). Thus, in the case of a bounded potential the coefficient of $\|\varphi_n\|$ in inequality (28) is of the order of $\lambda^{1/4}$ when $\lambda\to\infty$. This means that the length of the gaps $C(L)$ is of the order of $O(\lambda^{1/4})$, whereas in reality (see No. 31) the length of each such gap cannot exceed $2M$.

To improve inequality (28), it is possible to preserve the scheme of the proof of Theorem 3, changing only the smoothing factor in (24) so that the length of the smoothing interval be equal to some value $l=\frac{s}{3}(s=s(\mu))$ instead of $l=\frac{1}{3}$. For this purpose in (24) we replace $1-\theta(r-\rho_n)$ by $1-\theta\left(\frac{r-\rho_n}{l}\right)$, where the sequence $\rho_n\to\infty$ is chosen so that instead of (23) we have the inequality

$$J(\rho_n+l) < e^{2\mu l+\delta_1}J(\rho_n).$$

In this case the integration domain on the right-hand side of (26) is defined by the double-inequality $\rho_n+\frac{s}{3}\leqslant|OP|\leqslant\rho_n+\frac{2s}{3}$, and on the right-hand side of (27) — by the double inequality $\rho_n\leqslant|OP|\leqslant\rho_n+s$.

It is obvious that in (26) we now have $C_1=B_1s^{-1}$, $C_2=B_2s^{-2}$, and to pass from (26) to (27) one has to use the inequality (8) where $R_1-R=\frac{s}{3}$. It is easily seen from (7) that in this case we shall have in (8) $C=bs^{-2}+|q_0|+\lambda$. Inequality (27) thus takes the form

$$\|l[\varphi_n]-\lambda\varphi_n\|^2 < \left\{B_1^2s^{-2}\left(bs^{-2}+|q_0|+\lambda\right)+B_2^2s^{-4}\right\}(e^{2\mu s+\delta_1}-1)\|\varphi_n\|^2,$$

whence, setting $s = \mu^{-1}$, we obtain

$$\| L\varphi_n - \lambda\varphi_n \| < \gamma \| \varphi_n \|, \tag{29}$$

where

$$\gamma = 3\left\{B_1^2\mu^2\left(b\mu^2 + |q_0| + \lambda\right) + B_2^2\mu^4\right\}^{1/2}. \tag{30}$$

From relations (29) and (30) follows

Theorem 3bis. *Suppose for a given λ for some $\varphi(P;\lambda)$ the inequality*

$$|\varphi(P;\lambda)| < Ce^{\mu r}$$

is satisfied for all $P \in \mathfrak{G}_m$. Then the distance of the point λ from $S(L)$ does not exceed the value of γ given by formula (30). In particular, for small and for large μ this distance is of the order of μ and μ^2 respectively.

It is obvious that Theorem 3^{bis} significantly improves Theorem 3, which for small and for large μ gives for the distance from the spectrum a value of the order of $\sqrt{\mu}$ and $\exp \mu$ respectively. One application of Theorem 3^{bis} will be indicated at the end of No. 57.

55. The behavior of the solutions at points of the continuous part of the spectrum. It was established earlier in No. 53 that at isolated points of the spectrum the eigenfunctions decrease exponentially. In the present subsection, it will be shown for the one-dimensional case that at points λ of the continuous part of the spectrum the solutions $\varphi(x;\lambda)$ of the equation

$$-y'' + q(x)y = \lambda y \qquad (0 \leqslant x < \infty) \tag{31}$$

cannot, in general, increase rapidly.

It is established by the following lemma that the graph of the solution $y(x)$ of equation (31) cannot have high "peaks", i.e., for a radius δ of the neighborhood of the point x_0 which is sufficiently small and independent of x_0, the value of $|y(x)|$ cannot take on both sides of x_0 values too small as compared with $|y(x_0)|$. By its nature, the proof of this lemma is close to the proof of the well-known differential inequalities of S. A. Chaplygin.

Lemma 3 /108 (2)/. For a given value of λ, there exists a number $\delta > 0$ independent of x_0, such that for the solution $y(x) = \varphi(x;\lambda)$ of equation (31) we have in at least one of the two intervals $x_0 - \delta < x < x_0$, $x_0 < x < x_0 + \delta$,

$$|\varphi(x;\lambda)| \geqslant \frac{1}{2}|\varphi(x_0;\lambda)|.$$

Proof. Assuming for definiteness that $y(x_0) > 0$ and $y'(x_0) > 0$, we denote by $z(x)$ the solution of the equation

$$-z'' + q_0 z = \lambda z, \tag{32}$$

satisfying the initial conditions

$$z(x_0) = y(x_0), \quad z'(x_0) = y'(x_0).$$

From (31) and (32) follows the inequality

$$y''(x_0) - z''(x_0) = [q(x_0) - q_0]\, z(x_0) > 0,$$

so that in some right half-neighborhood of the point x_0 we have

$$y(x) > z(x).$$

Actually, the last inequality is valid for all $x > x_0$, since otherwise, letting x_1 denote the least value of x at which equality occurs ($x_1 > x_0$), $y(x_1) = z(x_1)$), we would obtain for some $x = x^*(x_0 < x^* < x_1)$ the equality $y'(x^*) = z'(x^*) = 0$,

contradicting the obvious relation

$$y'(x^*) - z'(x^*) = \int_{x_0}^{x^*} \{[q(x) - \lambda]\, y + [q_0 - \lambda]\, z\}\, dx > 0.$$

Without loss of generality, we may take $\lambda - q_0 > 0$, so that the graph of the solution $z(x)$ is sinusoidal with the period $2\pi(\lambda - q_0)^{-1/2}$. Therefore on the interval $(x_0,\ x_0 + \delta)$ for

$$\delta = \frac{\pi}{3\sqrt{\lambda - q_0}} \tag{33}$$

we have

$$y(x) > \frac{1}{2}\, y(x_0).$$

Q. E. D.

In the following lemma, $\rho(\lambda)$ denotes the spectral function of the operator L.

Lemma 4 /108 (2)/. If $f(x)$ is an arbitrary normalized function of $\mathfrak{L}_2(0,\ \infty)$ with a finite generalized Fourier transform $F(\lambda)$, so that

$$\int_0^\infty |f(x)|^2\, dx = 1 \tag{34}$$

and

$$f(x) = \int_{-a}^{a} F(\lambda)\, \varphi(x;\ \lambda)\, d\rho(\lambda), \tag{35}$$

then for a fixed finite value of a there exists a constant C independent of $f(x)$, such that

$$|f(x)| < C \qquad (0 \leqslant x < \infty).$$

Proof. Obviously, it is sufficient to find a number $\delta > 0$, independent of the function $f(x)$ and of x_0, such that for $|f(x_0)| > 1$ the inequality

$$|f(x)| \geqslant \frac{1}{4}\, |f(x_0)|$$

holds in at least one of the two intervals

Let x_0 be an arbitrary positive number. Assuming for definiteness $f(x_0) > 0$ and $f'(x_0) > 0$, we set

$$g(x) = -f''(x) + q(x)\, f(x) \tag{36}$$

and denote by $y(x)$ the solution of the differential equation

$$g(x) = -y''(x) + q_0 y(x)$$

with the initial condition

$$y(x_0) = f(x_0), \quad y'(x_0) = f'(x_0).$$

Since

$$f''(x_0) - y''(x_0) = [q(x_0) - q_0]\, f(x_0) > 0,$$

then in some right half-neighborhood of the point x_0 we have

$$f(x) > y(x). \tag{37}$$

Actually, the last inequality is valid for all $x > x_0$, since otherwise, letting x_1 denote the least value of x at which equality occurs ($(x_1 (x_1 > x_0)$, $f(x_1) = y(x_1)$), we would obtain for some $x = x^*$ $(x_0 < x^* < x_1)$ the equality $f'(x^*) = y'(x^*) = 0$, contradicting the obvious relation

$$f'(x^*) - y'(x^*) = \int_{x_0}^{x^*} \{q(x)\, f(x) - q_0 y(x)\}\, dx > 0.$$

Denoting the solution of the homogeneous equation

$$-z''+q_0z=0 \qquad (q_0<0)$$

with the initial conditions

$$z(x_0)=y(x_0), \quad z'(x_0)=y'(x_0),$$

by $z(x)$ we obtain

$$y(x)=z(x)+u(x), \tag{38}$$

where

$$u(x)=\frac{1}{\sqrt{|q_0|}}\int_{x_0}^{x}\sin\sqrt{|q_0|}(x-s)\,g(s)\,ds.$$

Since $z(x)$ is sinusoidal with the period $\frac{2\pi}{\sqrt{|q_0|}}$ and $z(x_0)>0$ and $z'(x_0)>0$, then in the interval $(x_0,\ x_0+\delta_1)$ for $\delta_1=\frac{\pi}{3\sqrt{|q_0|}}$ we have

$$z(x)>\frac{1}{2}z(x_0). \tag{39}$$

For $x>x_0$ the second term on the right-hand side of (38) does not exceed in absolute value

$$\frac{x-x_0}{\sqrt{|q_0|}}\left\{\int_{x_0}^{x}|g(s)|^2\,ds\right\}^{1/2}<\frac{x-x_0}{\sqrt{|q_0|}}\|g\|,$$

but by virtue of (34), (35) and (36) we have

$$\|g\|^2=\left\|\int_{-a}^{a}\lambda F(\lambda)\varphi(x;\lambda)\,d\rho(\lambda)\right\|^2=\int_{-a}^{a}|\lambda F(\lambda)|^2\,d\rho(\lambda)\leqslant a^2,$$

so that

$$u(x)\leqslant\frac{a(x-x_0)}{\sqrt{|q_0|}} \qquad (x>x_0). \tag{40}$$

For $x-x_0<\frac{\sqrt{|q_0|}}{4a}$ we have

$$|y(x)-z(x)|<\frac{1}{4}<\frac{1}{4}f(x_0)$$

and consequently, for

$$\delta=\min\left\{\delta_1,\ \frac{\sqrt{|q_0|}}{4a}\right\}$$

we find for $x_0<x<x_0+\delta$ by virtue of (37), (39) and (40)

$$f(x)>y(x)=z(x)+u(x)>\frac{1}{2}z(x_0)+u(x)=$$

$$=\frac{1}{2}f(x_0)+u(x)>\frac{1}{4}f(x_0).$$

Q. E. D.

Theorem 6 /108 (2)/. If the inequality $q(x)>q_0>-\infty$ is satisfied for $x\geqslant 0$, then for any $\varepsilon>0$ and for almost all the values of λ, with respect to

the spectral measure $\rho(\lambda)$, *a constant* $C_\varepsilon(\lambda)$ *is found, such that for* $x \geqslant 0$ *we have*

$$|\varphi(x;\lambda)| < C_\varepsilon(\lambda)\,|x|^{\frac{1}{2}+\varepsilon} \tag{41}$$

Proof. Let $F(\lambda)$ be an arbitrary function of the Hilbert space $\mathfrak{L}^2_\rho(-a, a)$ in which the symbol of the scalar product and that of the norm are suffixed with the letter ρ.

By virtue of Lemma 2 we have

$$\left|\int_{-a}^{a} F(\lambda)\,\varphi(x;\lambda)\,d\rho(\lambda)\right| < C\,\|F\|_\rho,$$

i.e.,

$$|(F,\ \varphi)_\rho| < C\,\|F\|_\rho,$$

whence, since the function $F \in \mathfrak{L}^2_\rho(-a,\ a)$ is arbitrary, it follows for any $x \geqslant 0$ that

$$\|\varphi\|^2_\rho \leqslant C,$$

that is

$$\int_{-a}^{a} |\varphi(x;\lambda)|^2\,d\rho(\lambda) \leqslant C \qquad (x \geqslant 0). \tag{42}$$

Setting $\lambda = a$ in formula (33), we split up the semi-axis $x \geqslant 0$ into intervals of length

$$\delta = \frac{\pi}{3\sqrt{a - q_0}}$$

and denote the middle points of these intervals by x_n $(n = 1, 2, \ldots)$, so that, by virtue of Lemma 1, for $x_n < x < x_{n+1}$ we have either

$$\left.\begin{aligned} &|\varphi(x;\lambda)| < 2\,|\varphi(x_n;\lambda)|, \\ \text{or}\quad &|\varphi(x;\lambda)| < 2\,|\varphi(x_{n+1};\lambda)|. \end{aligned}\right\} \tag{43}$$

Moreover, from condition (42) it follows that the ρ-measure of the set of those λ which for some $\varepsilon > 0$ and given value of n satisfy

$$|\varphi(x_n;\lambda)| > C x_n^{\frac{1}{2}+\varepsilon} \tag{44}$$

is smaller than $1/Cx^{1+2\varepsilon}$, i.e., smaller than $1/C_1 n^{1+2\varepsilon}$. The ρ-measure of the set of those λ which for some $\varepsilon > 0$ and for all $n \geqslant m$ satisfy (44), is therefore smaller than

$$\eta_m = \frac{1}{C_1} \sum_{n=m}^{\infty} \frac{1}{n^{1+\varepsilon}}. \tag{45}$$

Thus, for $m \geqslant n$ for a given $\varepsilon > 0$ and for all λ, with the exception of a set of ρ-measure smaller than η_m, we have the inequality

$$|\varphi(x_n;\lambda)| < C x_n^{\frac{1}{2}+\varepsilon},$$

and consequently, also the inequality

$$|\varphi(x_n; \lambda)| < C_2 x_n^{\frac{1}{2}+\varepsilon} \qquad (n = 1, 2, \ldots).$$

Hence, using inequalities (43), we obtain for a certain value $C_m = C$ and for some $\varepsilon \geqslant 0$, for all $x \geqslant 0$ and for all λ, with the exception of a set of ρ-measure smaller than η_m, the inequality

$$|\varphi(x; \lambda)| < C x^{\frac{1}{2}+\varepsilon} \qquad (0 \leqslant x < \infty). \tag{46}$$

Since, by virtue of (45), $\eta_m \to 0$ for $m \to \infty$, it then follows from the above inequality that for some value $C_\varepsilon(\lambda)$ of the constant, relation (46) is satisfied for almost all the λ, in the sense of the ρ-measure, Q.E.D.

By other methods, Yu.M. Berezanskii, G.I. Kats and A.G. Kostyuchenko (see /11/) succeeded in generalizing Theorem 6 to the multi-dimensional case. The exponent on the right-hand side of inequalities (41)' is then $\frac{m}{2}+\varepsilon$, where $m \geqslant 1$ is the dimension of the Euclidean space spanned by the points P.

In connection with the behavior of the solutions at points of the continuous part of the spectrum, see also No. 58.

56. The spectrum of the Hill differential operator. In order to apply the above general results to the investigation of the spectrum of the one-dimensional two-term differential operator with a periodic potential, we shall first give the fundamentals of the theory of the Hill equation.

The Hill equation has the form

$$-y'' + q(x)\, y = \lambda y, \tag{47}$$

where $q(x)$ is a continuous periodical function, defined on the whole axis. Without loss of generality, we may consider the period equal to unity, so that for all x

$$q(x+1) = q(x).$$

In this subsection we call a function p e r i o d i c a l if it has a period equal to one, and s e m i - p e r i o d i c a l if as the argument varies by one the function is multiplied by minus one.

We denote by $C(x; \lambda)$ and $S(x; \lambda)$ the solutions of equation (47) satisfying the initial conditions

$$\begin{aligned} C(0; \lambda) &= 1, \quad C'(0; \lambda) = 0, \\ S(0; \lambda) &= 0, \quad S'(0; \lambda) = 1, \end{aligned} \tag{48}$$

so that the general solution has the form

$$y(x; \lambda) = \alpha C(x; \lambda) + \beta S(x; \lambda). \tag{49}$$

Also

$$y(x+1; \lambda) = \alpha C(x+1; \lambda) + \beta S(x+1; \lambda). \tag{50}$$

Since the functions $C(x+1; \lambda)$ and $S(x+1; \lambda)$, are obviously solutions of equation (47) too, then

$$\begin{aligned} C(x+1; \lambda) &= a_{11} C(x; \lambda) + a_{21} S(x; \lambda), \\ S(x+1; \lambda) &= a_{12} C(x; \lambda) + a_{22} S(x; \lambda). \end{aligned} \tag{51}$$

To obtain the general solution of equation (47) in the form of Floquet /3/, we look for an auxiliary solution (49) of this equation, satisfying the condition

$$y(x+1;\ \lambda)=\rho y(x;\ \lambda). \tag{52}$$

Substituting (51) in (50) and using (52) we obtain a homogeneous linear system of equations for the constants α and β in (50). This system admits a non-trivial solution if and only if

$$\begin{vmatrix} a_{11}-\rho & a_{12} \\ a_{21} & a_{22}-\rho \end{vmatrix}=0. \tag{53}$$

From (48) and (51) it follows that

$$a_{11}=C(1;\ \lambda),\quad a_{12}=S(1;\ \lambda),$$
$$a_{21}=C'(1;\ \lambda),\quad a_{22}=S'(1;\ \lambda),$$

so that equation (53) takes the form

$$\rho^2-[C(1;\ \lambda)+S'(1;\ \lambda)]\rho+1=0.$$

Introducing the entire transcendental function of the parameter λ

$$A(\lambda)=\frac{1}{2}[C(1;\ \lambda)+S'(1;\ \lambda)], \tag{54}$$

we obtain the equation

$$\rho^2-2A(\lambda)\rho+1=0, \tag{55}$$

from which the so-called multiplicators are determined

$$\rho_{1,2}=A(\lambda)\pm\sqrt{A^2(\lambda)-1}. \tag{56}$$

If for a given λ the discriminant $A^2(\lambda)-1$ is not equal to zero, then equation (47) admits two linearly independent solutions satisfying the condition (52), $y_1(x;\ \lambda)$ and $y_2(x;\ \lambda)$. It is obvious that the functions

$$\varphi_1(x;\ \lambda)=e^{-\mu_1 x}y_1(x;\ \lambda),$$
$$\varphi_2(x;\ \lambda)=e^{-\mu_2 x}y_2(x;\ \lambda),$$

where

$$e^{\mu_{1,2}}=\rho_{1,2}. \tag{57}$$

are periodical, whence follows the representation of the general solution of the Hill equation in Floquet's form

$$y(x;\ \lambda)=C_1e^{\mu_1 x}\varphi_1(x;\ \lambda)+C_2e^{\mu_2 x}\varphi_2(x;\ \lambda). \tag{58}$$

A real value of the parameter λ is called stable* if all the solutions of equation (47) remain bounded for $|x|\to\infty$, and unstable** in the opposite case.

If the multiplicators are real and distinct, it follows from Floquet's formula (58) that the given value of the parameter λ is unstable. If the multiplicators are not real, then they are complex conjugate, and since from (56) their product is equal to one, $|\rho_{1,2}|=1$ Thus in (57) the characteristic exponents $\mu_{1,2}$ are imaginary and it follows from formula (58) that this value of the parameter λ is stable.

* [Or, point of stability.]
** [Or, point of instability.]

Let $\rho_1 = \rho_2 = \rho$. Then, according to (56), we have $\rho = \pm 1$, and from (52), one of the solutions of equation (47) is periodical (for $\rho = 1$) or semi-periodical (for $\rho = -1$). If for $\rho_1 = \rho_2$ both solutions are periodical (or semi-periodical), then obviously, the given value of the parameter λ is stable. In the opposite case the second solution $y_2(x; \lambda)$ satisfies the relation

$$y_2(x+1; \lambda) = \rho[y_2(x; \lambda) + y_1(x; \lambda)], \tag{59}$$

where the solution $y_1(x, \lambda)$ satisfies condition (52). Obviously, the solution $y_2(x, \lambda)$ is unbounded, so that the corresponding value of the parameter λ is unstable.

It can easily be seen that for negative λ with large absolute values, equation (47) has unbounded solutions. Therefore, the set of points of instability [the instability domain] always contains some semi-axis $(-\infty; \lambda_0)$.

In particular, if $q(x) \equiv 0$ the instability domain coincides with the semi-axis $\lambda \leqslant 0$, and the stability domain coincides with the semi-axis $\lambda > 0$. In order to elucidate the structure of the stability domain in the general case we introduce the two boundary-value problems generated in $\mathcal{L}_2(0, 1)$ by equation (47), having the boundary conditions

$$y(0) = y(1), \qquad y'(0) = y'(1) \tag{60}$$

and

$$y(0) = -y(1), \quad y'(0) = -y'(1). \tag{61}$$

respectively.

It follows from (48) and (54) that the spectrum of the periodical boundary-problem (47), (60) coincides with the set of the roots of the equation

$$A(\lambda) - 1 = 0,$$

and the spectrum of the semi-periodical boundary-value problem (47), (61) coincides with the set of the roots of the equation

$$A(\lambda) + 1 = 0.$$

On the other hand, as it was shown already by A. M. Lyapunov (see, for example, /53 (4)/, and also /3/), the spectra $\{\lambda'_k\}_{k=0}^{\infty}$ and $\{\lambda''_k\}_{k=1}^{\infty}$ of the periodical boundary-value problems are related by the inequalities

$$\lambda'_0 < \lambda''_1 \leqslant \lambda''_2 < \lambda'_1 \leqslant \lambda'_2 < \lambda''_3 \leqslant \lambda''_4 < \lambda'_3 \leqslant \lambda'_4 < \ldots$$

From the above one easily concludes /3/ that the stability domain consists of the open intervals (the stability zones)

$$(\lambda'_0, \lambda''_1), \quad (\lambda''_2, \lambda'_1), \quad (\lambda'_2, \lambda''_3), \ldots, \tag{62}$$

and the instability domain consists of the semi-axis $(-\infty, 0]$ and of the closed intervals (the instability zones)

$$[\lambda''_1, \lambda''_2], \quad [\lambda'_1, \lambda'_2], \ldots \tag{63}$$

If some eigenvalue of a periodical or semi-periodical problem is multiple, the corresponding segment of the system (63) reduces to a point. Since this λ-point is stable, the two neighboring intervals of (62) bounded on one of the sides by this point merge into one stability zone. In particular if $q(x) \equiv 0$ all the eigenvalues of the periodical and semi-periodical problem

are double and therefore all the zones (62) merge into one stability zone [the semi-axis] $\lambda > 0$.

Let the operator L be defined in $\mathfrak{L}_2(-\infty, \infty)$ by the Hill operation

$$-y'' + q(x)\,y \qquad [q(x+1) = q(x)].$$

By virtue of Theorem 34 of No. 16, the Hill operator L is self-adjoint. The following theorem characterizes the spectrum of the operator L.

Theorem 7 (see /19 (2)/). The continuous part of the spectrum of the Hill operator L consists of the stability zones and the boundary points of the instability zones. The discrete part of the spectrum of the operator L is the empty set, so that the interior points of the instability zones are regularity points of the operator L.

Proof. We first show that $D(L) = \emptyset$.

If $\lambda \in D(L)$ then from Theorem 1 of No. 53 the corresponding eigenfunction tends to zero for $|x| \to \infty$, but from Floquet's formulas (58) and (59) it follows that no such solutions exist for any real value of the parameter λ.

Therefore $D(L) = \emptyset$, so that the second part of the theorem is proved.

Since the set $C(L)$ is closed, to prove the first part of the theorem it is sufficient to show that the points of any stability zone belong to $C(L)$ and that no interior point of some instability zone can belong to $C(L)$.

If the point λ belongs to a stability zone, then the corresponding solution (58) is bounded for $|x| \to \infty$. Hence, on the basis of Theorem 5 of No. 54 it follows that $\lambda \in S(L)$. Since $D(L) = \emptyset$, then $\lambda \in C(L)$.

If λ is any interior point of an instability zone, then the solution $y(x; \lambda)$ is not only unbounded for $|x| \to \infty$, but it does not even satisfy the estimate (41). Hence, on the basis of Theorem 6 we conclude that the spectral measure of the open instability zone under consideration is equal to zero; consequently there are no points of the spectrum $S(L)$ inside this zone. Thus for any interior point λ of each instability zone we have $\lambda \notin C(L)$. The theorem is now completely proved.

The spectrum of the Hill operator is therefore purely continuous and has the gaps (63).

Another method for determining the spectrum and obtaining the inversion formulas for the one-dimensional and multi-dimensional Hill operators is due to I. M. Gel'fand /27 (1)/ (see also V. B. Lidskii's supplement /95 (2)/).

From the considerations of this subsection it follows that the problem of the asymptotic behavior of the lengths of the gaps (63) reduces to finding the asymptotic formulas for the eigenvalues of the periodical and semi-periodical Sturm-Liouville boundary-value problem on a finite interval. It can be shown by the usual methods that if a periodical potential $q(x)$ has r continuous derivatives, then for large λ the length of the gap centered at the point λ is $o(\lambda^{-r/2})$. If nothing is known about the differential properties of the function $q(x)$, it is impossible to guarantee a specific rate of decrease of the gap lengths. Nevertheless it is still possible to state that the gap lengths tend to zero for $\lambda \to \infty$ (see, for example, / 95 (2)/, where the function $q(x)$ is assumed only piecewise continuous).

The number of gaps in the sequence (63) is not necessarily infinite. Aside from the trivial case $q(x) \equiv 0$ for which there are no gaps at all in $C(L)$ there exist non-constant periodical potentials with a finite number of gaps in the sequence (63). In a recent investigation of N.I. Akhiezer /6 (5)/ from a given gap (rather, from a specific form of the spectral function $\rho(\lambda)$, constant for $\lambda < 0$ and in a given interval of the semi-axis $\lambda > 0$, but increasing at all the remaining points of the λ-axis) a periodical potential is constructed, for which the given gap is the only one in $C(L)$.

We do not investigate here the problem of the character of the discrete spectrum appearing in the gaps of (63) when the periodical potential is perturbed by a perturbation $\eta(x)$ infinitely small for $|x| \to \infty$. Some examples of the set of eigenvalues introduced by perturbations in the gaps of the spectra of other operators may be found further on in Nos. 62 and 67.

The character of the spectrum introduced in the first, semi-infinite, gap $(-\infty, a_1)$ by perturbations $\eta(x)$ of a periodical potential $q(x)$ can easily be investigated by the methods of No. 30. For this purpose we assume without loss of generality that $a_1 = 0$. We transform the quadratic functional

$$\Phi[y] = \int_{-\infty}^{\infty} [y'^2 + q(x)\, y^2 + \eta(x)\, y^2]\, dx$$

by the substitution $y = \varphi(x)\, z$ /13 (7)/, where $\varphi(x)$ is the first eigenfunction of the boundary-value problem (47), (60). Using the relation resulting from integrating the identity

$$[\varphi'' - q(x)\varphi(x)]\, \varphi(x)\, y^2 \equiv 0$$

over the axis $(-\infty, \infty)$, with any finite function $y \in \mathfrak{D}_L$, we then obtain for the functional $\Phi[y]$ the expression

$$\Phi[y] = \int_{-\infty}^{\infty} \varphi^2(x)\, [z'^2 + \eta(x)\, z^2]\, dx.$$

Furthermore, since the periodical function $\varphi(x)$ satisfies the double inequality

$$0 < m \leqslant \varphi(x) \leqslant M < \infty,$$

the factor $\varphi^2(x)$ in the latter integrand does not play a significant role. Therefore a number of results established in No. 30 in the investigation of the functional

$$F[z] = \int_{-\infty}^{\infty} [z'^2 + \eta(x)\, z^2]\, dx,$$

remain valid for $\Phi[y]$. In particular, Theorem 15 of No. 30 holds for the operator L_η generated by the perturbed Hill operation

$$-y'' + q(x)\, y + \eta(x)\, y,$$

provided we set $n = 1$ and $q(x) = \eta(x)$.

57. The Hartman-Putnam theorem on the gaps in the spectrum of operators with a bounded potential. Let L be an operator generated in $\mathfrak{L}_2(0, \infty)$ by the operation

$$-y'' + q(x)\, y$$

with the bounded potential

$$|q(x)| < M. \tag{64}$$

According to Theorem 17 of No. 31, the length of any gap in $C(L)$ does not exceed M. However, similarly to the case of the Hill operation, the lengths of the gaps in $C(L)$ which are remote from the point $\lambda = 0$ may, under certain additional restrictions on the potential, become as small as we please. These subtle results are due to Hartman and Putnam /103 (2)/.

The following theorem is one of the results due to these authors.

Theorem 8 /103 (2)/. If the function $q(x)$ is bounded and uniformly continuous on the semi-axis $x \geqslant 0$, then the length of the gap in $C(L)$ situated to the right of the point λ tends to zero for $\lambda \to \infty$.

Proof. Let $\varepsilon > 0$ be arbitrarily given. It is required to find a number λ such that for any $\lambda > \lambda_\varepsilon$ the interval $(\lambda, \lambda + \varepsilon)$ includes points of $C(L)$. On the basis of Theorem 26 of No. 9 it is sufficient to establish the limit relation

$$\lim_{t \to \infty} [N(t, \lambda + \varepsilon) + N(t, \lambda)] = \infty$$

for any $\lambda > \lambda_0$, or by virtue of the well-known relation between $N(t, \lambda)$, the number of points of the spectrum of the regular boundary-value problem situated to the left of a given bound λ, and $\nu(t, \lambda)$, the number of the zeros of the solution of equation (31) in the interval $(0, t)$, it is sufficient to establish

$$\lim_{t \to \infty} [\nu(t, \lambda + \varepsilon) - \nu(t, \lambda)] = \infty. \tag{65}$$

Taking the number λ so large that it satisfies

$$\lambda - M > \frac{\lambda}{4}, \tag{66}$$

the solutions $y(x; \lambda)$ of equation (31) oscillate. We denote the successive zeros of the function $y(x; \lambda)$ by x_k $(k = 1, 2, \ldots)$.

Furthermore, since the roots of the functions $y(x; \lambda)$ and $y'(x; \lambda)$ alternate, then as x increases from x_k to x_{k+1} the fraction

$$\frac{y'(x; \lambda)}{\sqrt{\lambda}\, y(x; \lambda)}$$

varies continuously from $+\infty$ to $-\infty$. Hence it follows that to within $O(1)$ the function $\nu(x, \lambda)$ is equal to

$$-\operatorname{arctg} \frac{y'}{\sqrt{\lambda}\, y} = \theta(x, \lambda), \tag{67}$$

and (65) is equivalent to the limit relation

$$\lim_{t \to \infty} [\theta(t, \lambda + \varepsilon) - \theta(t, \lambda)] = \infty. \tag{68}$$

From (67) and (31) one easily obtains

$$\theta'(x, \lambda) = \sqrt{\lambda} + \frac{1}{\sqrt{\lambda}} q(x) \frac{\lambda y^2(x; \lambda)}{\lambda y^2(x; \lambda) + y'^2(x; \lambda)},$$

whence

$$\theta(t, \lambda) = \sqrt{\lambda}\, t + \frac{1}{\sqrt{\lambda}} \int_0^1 q(x) \frac{\lambda y^2(x; \lambda)\, dx}{\lambda y^2(x; \lambda) + y'^2(x; \lambda)} + \theta(0, \lambda). \tag{69}$$

In this representation the homogeneity of the coefficient (multiplier) of $q(x)$ is essential, since in further estimates it enables us not to use the same solution $y(x; \lambda)$ under the integral in (69) when t increases, but to replace $y(x; \lambda)$ in each interval (x_k, x_{k+1}) by any other solution $y_k(x; \lambda)$ of equation (31), satisfying the condition $y_k(x_k; \lambda) = 0$.

If the number μ_k is determined by the equality

$$x_{k+1} - x_k = \frac{\pi}{\sqrt{\mu_k}}, \tag{70}$$

then we have by virtue of Sturm's comparison theorem,

$$\lambda - \max_{x_1 \leqslant x \leqslant x_2} q(x) \leqslant \mu \leqslant \lambda - \min_{x_1 \leqslant x \leqslant x_2} q(x),$$

so that

$$\mu_k = \lambda - q(\xi_k) \tag{71}$$

where $x_k \leqslant \xi_k \leqslant x_{k+1}$. Now, one of the solutions $y_k(x; \lambda)$ of equation (31), vanishing for $x = x_k$, obviously satisfies the integral equation

$$y_k(x; \lambda) = \sin \sqrt{\mu_k}(x - x_k) + \frac{1}{\sqrt{\mu_k}} \int_{x_k}^{x} Q_k(s)\, y_k(s; \lambda) \sin \sqrt{\mu_k}(x - s)\, ds. \tag{72}$$

where

$$Q_k(s) = q(s) - q(\xi_k). \tag{73}$$

Using the integral equation (72), we find an estimate of the solution $y_k(x; \lambda)$ on the interval $x_k \leqslant x \leqslant x_{k+1}$. Setting

$$m_k = \max_{x_k \leqslant x \leqslant x_{k+1}} |y_k(x; \lambda)|$$

and taking relations (70), (73) and (64) into account, we obtain

$$m_k \leqslant 1 + \frac{2\pi M m_k}{\mu_k},$$

so that for sufficiently large λ

$$m_k \leqslant (1 - 2\pi M \mu_k^{-1}) \leqslant 2. \tag{74}$$

since, by virtue of (71), μ_k together with λ is also sufficiently large. Using again the integral equation (72) and relation (74), we obtain

$$\begin{aligned} y_k(x; \lambda) &= \sin \sqrt{\mu_k}(x - x_k) + O\left(\frac{1}{\lambda}\right); \\ y_k'(x; \lambda) &= \sqrt{\mu_k} \cos \sqrt{\mu_k}(x - x_k) + O\left(\frac{1}{\sqrt{\lambda}}\right), \end{aligned} \tag{75}$$

uniformly with respect to k, for $\lambda \to \infty$.

Moreover, it follows from (71) that

$$|\sqrt{\mu_k} - \sqrt{\lambda}| = O\left(\frac{1}{\sqrt{\lambda}}\right),$$

so that

$$y_k'(x; \lambda) = \sqrt{\lambda} \cos \sqrt{\mu_k}(x - x_k) + O\left(\frac{1}{\sqrt{\lambda}}\right). \tag{76}$$

From the asymptotic formulas (75) and (76) we obtain for the coefficient $q(x)$ in (69) the relation

$$\frac{\lambda y^2(x; \lambda)}{\lambda y^2(x; \lambda) + y'^2(x; \lambda)} = \sin^2 \sqrt{\mu_k}(x - x_k) + O\left(\frac{1}{\lambda}\right), \tag{77}$$

valable on the interval $x_k \leqslant x \leqslant x_{k+1}$ for $\lambda \to \infty$.

Now, using the asymptotic formula (77), we show that the integral in the representation (69), divided by t, i.e., the quantity

$$\frac{1}{t} \int_0^t q(x) \frac{\lambda y^2(x; \lambda)\, dx}{\lambda y^2(x; \lambda) + y'^2(x; \lambda)} \tag{78}$$

can be represented as a sum of two terms, the first of which is, independent of λ, and the second tends to zero, uniformly with respect to t, for $\lambda \to \infty$.

For this purpose, representing the right-hand side of (77) in the form

$$\frac{1}{2} - \frac{1}{2} \cos 2\sqrt{\mu_k}(x - x_k) + O\left(\frac{1}{\lambda}\right)$$

and separating from (78) the term

$$\frac{1}{2t} \int_0^t q(x)\, dx, \tag{79}$$

independent of λ, we show that for $\lambda \to \infty$ the integral

$$\frac{1}{t}\int_0^t q(x)\cos 2\sqrt{\mu_k}(x-x_k)\,dx \tag{80}$$

tends to zero uniformly with respect to t. In this connection, we make use of the uniform continuity of the function $q(x)$ on the semi-axis $x \geqslant 0$, assumed in the statement of the theorem.

The integration interval $(0, t)$ in (80) may be replaced by the interval (x_1, x_n) where $x_n \leqslant t < x_{n+1}$, since, from relations (70) and (71), the error involved tends to zero uniformly with respect to t as $\lambda \to \infty$. Furthermore, since (70) implies

$$\int_{x_k}^{x_{k+1}} \cos 2\sqrt{\mu_k}(x-x_k)\,dx = 0 \qquad (k = 1, 2, \ldots),$$

the function $q(x)$ may be replaced in (80) by the function $q(x) - q(x_k)$, where $x_k \leqslant x < x_{k+1}$. The quantity (80) then reduces to the expression

$$\frac{1}{t}\sum_{k=1}^{n}\int_{x_k}^{x_{k+1}} [q(x)-q(x_k)]\cos 2\sqrt{\mu_k}(x-x_k)\,dx,$$

which is less in absolute value than the quantity

$$\frac{1}{t}\sum_{k=1}^{n}\int_{x_k}^{x_{k+1}} \omega_k\,dx = \frac{1}{t}\sum_{k=1}^{n}(x_{k+1}-x_k)\,\omega_k, \tag{81}$$

where ω_k is the oscillation of the function $q(x)$ in the interval (x_k, x_{k+1}). It follows from (70) and (71) that for sufficiently large λ the length of the interval $x_{k+1} - x_k$ becomes as small as we please. Thus each oscillation ω_k $(k = 1, 2, \ldots)$ is also as small as we please. Consequently, the average oscillation (81) may be made as small as we please, uniformly with respect to t.

The representation (69) now takes the form

$$\theta(t, \lambda) = t\left[\sqrt{\lambda} + \frac{1}{2\sqrt{\lambda}t}\int_0^t q(x)\,dx + \frac{1}{\sqrt{\lambda}}a(t, \lambda)\right], \tag{82}$$

where

$$\lim_{\lambda\to\infty} a(t, \lambda) = 0, \tag{83}$$

uniformly with respect to t.

Together with (82) we have the formula

$$\theta(t, \lambda+\varepsilon) = t\left[\sqrt{\lambda+\varepsilon} + \frac{1}{2\sqrt{\lambda+\varepsilon}\,t}\int_0^t q(x)\,dx + \frac{1}{\sqrt{\lambda+\varepsilon}}a(t, \lambda+\varepsilon)\right].$$

so that for the difference we have

$$\theta(t, \lambda+\varepsilon)-\theta(t, \lambda)=$$

$$=t\left\{\frac{\varepsilon}{2\sqrt{\lambda}}-\frac{\varepsilon}{4\sqrt{\lambda^3}t}\int_0^t q(x)\,dx-\frac{\varepsilon}{2\sqrt{\lambda^3}}a(t, \lambda+\varepsilon)+\frac{1}{\sqrt{\lambda}}[a(t, \lambda+\varepsilon)-a(t, \lambda)]+\ldots\right\},$$

where the points represent small quantities of order higher than that of ε. The second and third terms in the braces may be made small in comparison with the first term by choosing λ sufficiently large.

If we then take the limits for $t\to\infty$, the fourth term in the braces tends to zero by virtue of (83). Thus, however small the number $\varepsilon>0$, for all values of λ larger than some λ_ε we have the limit relation (68). The theorem is thus completely proved.

In the cited paper of Hartman and Putnam /103 (2)/, together with the above theorem, further results regarding the asymptotic behavior of the gaps for $\lambda\to\infty$ were obtained. The meaning of these results is that under the assumption of boundedness and uniform continuity of $q'(x)$, the length of the gaps decreases faster than $1/\sqrt{\lambda}$, and under the assumption of boundedness and uniform continuity of $q'(x)$ and $q''(x)$—faster than $1/\lambda$.

A similar result can be obtained by means of Theorem 3bis of No. 54. Indeed, by means of the transformation indicated in /61 (1), (page 62)/, it can easily be shown that when the potential $q(x)$ has two uniformly bounded derivatives, for any solution of equation (31) we have the estimate

$$|\varphi(x; \lambda)|<A\exp M\lambda^{-1/2}x.$$

Thus, in the statement of Theorem 3bis we have $\mu=\lambda^{-1/2}$, and then relation (30) of No. 54 gives $\gamma=O(\lambda^{-1})$.

However, it is still unclear whether the assumption of uniform boundedness of higher derivatives of the potential will lead to a further decrease of the upper bound of the gap lengths, as in the case of periodical potentials. On the other hand, it is not certain that the requirement of uniform continuity of the potential is essential for the validity of Theorem 8.

In their electronic theory of metals /44/ Sommerfeld and Bethe argue that in the case of a periodical potential the decrease in the gaps in the continuous spectrum of the energy is due to the fact that when the total energy increases $\lambda\to\infty$ the effect of the bounded potential energy $q(x)$ should become smaller and smaller, so that the behavior of the particle should approach that of a free particle. It is of interest to check this argument for any bounded potential.

58. **On V. A. Steklov's problem in the theory of orthogonal polynomials and its continual analog.** Let $\{P_n(\lambda)\}_{n=0}^{\infty}$ be the system obtained from the sequence $1, \lambda, \lambda^2, \ldots$ by means of the Sonine-Schmidt orthogonalization process in the metric of $\mathfrak{L}_\sigma^2(0, 1)$, so that

$$\int_0^1 P_m(\lambda)\overline{P_n(\lambda)}\,d\sigma(\lambda)=\delta_{mn}. \tag{84}$$

For any function $f(\lambda)\in\mathfrak{L}_\sigma^2(0, 1)$ we have in the sense of convergence in $\mathfrak{L}_\sigma^2(0, 1)$,

$$f(\lambda)=\sum_{n=0}^{\infty}c_nP_n(\lambda), \tag{85}$$

where

$$c_n=\int_0^1 f(\lambda)\overline{P_n(\lambda)}\,d\sigma(\lambda).$$

One of the basic problems of the theory of orthogonal polynomials is the investigation of the Fourier-Chebyshev series (85) for convergence at

a point or uniform convergence. It then turns out that the classical theorems on the convergence of the Fourier series are carried over to the case of expansions in a Fourier-Chebyshev series, provided the polynomials forming the orthonormalized system are bounded.

In this connection, in 1921 V. A. Steklov /92/ formulated the problem of the boundedness conditions of an orthonormalized system of polynomials in the whole orthogonality range or in part of it (in terms of weight). Confining himself to absolutely continuous functions $\sigma(\lambda)$. V. A. Steklov conjectured the uniform boundedness of any orthogonal system of polynomials in each segment $[\alpha, \beta]$, lying inside the orthogonality interval $(0, 1)$ and not containing zeros of the weight function $p(\lambda) = \sigma'(\lambda)$. All the known systems of polynomials, from the Legendre polynomials to the polynomials recently constructed by N. I. Akhiezer /6 (3)/, orthogonal over a system of segments, justify V. A. Steklov's hypothesis. However, the problem of whether this hypothesis is correct remains open.

In his monograph /30 (2)/, which is largely devoted to the problem of V. A. Steklov, Ya. L. Geronimus established that in the case of uniform boundedness of any infinite subsequence $\{P_{n_k}(\lambda)\}$ on the segment $[\alpha, \beta]$ we have $p(\lambda) \geqslant m > 1$, or if we do not assume the function $\sigma(\lambda)$ absolutely continuous,

$$\sigma(\mu) - \sigma(\lambda) \geqslant m(\mu - \lambda) \quad (\alpha < \lambda < \mu < \beta;\ m > 0). \tag{86}$$

Thus the condition assumed by V. A. Steklov for the [uniform] boundedness of the system $\{P_n(\lambda)\}$ [(i.e., the non-vanishing of $p(\lambda)$ on the segment) is necessary.

In a wider sense, it is natural to refer to V. A. Steklov's problem all problems where it is necessary to determine the behavior of the sequence $\{P_n(\lambda)\}$ for $n \to \infty$ from the properties of the function $\sigma(\lambda)$. In particular, the following theorem of Ya. L. Geronimus /30 (2)/ belongs to this class of problems: if condition (86) is satisfied, then for a fixed, arbitrarily small $\varepsilon > 0$, the averages

$$\frac{1}{n+1} \sum_{k=0}^{n} |P_k(\lambda)|^2 \tag{87}$$

will be bounded uniformly on the segment $[\alpha + \varepsilon, \beta - \varepsilon]$ with respect to n and λ. The proof of this theorem is based on a generalization of the results of G. Szegö /86/ and on the use of a special system of polynomials, constructed by N. I. Akhiezer /6 (1)/ in connection with some problems of aerodynamics.

The problem of V. A. Steklov can be treated as one of the problems of the qualitative spectral analysis of the Jacobi matrices. This problem is included in the program of the present chapter, since by virtue of a recurrence relation, the polynomial $P_n(\lambda)$ as a function of n is a solution of a finite-difference equation of the form

$$-\alpha_{n+1} y_{n+1} + \beta_n y_n - \alpha_n y_{n-1} = \lambda y_n. \tag{88}$$

We have in mind the relation between the behavior of the spectral function of the Jacobi matrix at the point λ and the behavior of the solution of equation (88) for $n \to \infty$.

Owing to the known connections between infinite Jacobi matrices and the singular Sturm-Liouville problem (see, for example, /6 (4)/), an analogy should exist between the behavior of the solution of equation (88)

for $n \to \infty$ and of that of equation (31) of No. 55 for $x \to \infty$, depending on the properties of the spectral function in the neighborhood of the point λ. In particular, the following theorem, due to E. E. Shnol', is the discrete analog of Theorem 6 of No. 55.

Theorem 9 /108 (2)/. For any $\varepsilon > 0$ and for almost all the values of λ, with respect to the spectral measure $\sigma(\lambda)$, we have the bound

$$|P_n(\lambda)| \leqslant C(\lambda; \varepsilon) n^{\frac{1}{2}+\varepsilon}. \tag{89}$$

Proof. By virtue of relation (74) for $m = n$, the σ-measure of the set M_n of points λ which for fixed C and $\varepsilon > 0$ satisfy

$$|P_n(\lambda)| \geqslant C n^{\frac{1}{2}+\varepsilon}, \tag{90}$$

cannot exceed

$$\frac{1}{C^2 n^{1+2\varepsilon}}.$$

Therefore the σ-measure of the set $M = \sum_{n=1}^{\infty} M_n$ of points λ at which for given C and $\varepsilon > 0$ inequality (90) is valid at least for one of the values $n = 1, 2, \ldots$, does not exceed

$$\frac{1}{C^2} \sum_{n=1}^{\infty} \frac{1}{n^{1+2\varepsilon}}.$$

Thus, on the complementary set, whose σ-measure is not less than

$$1 - \frac{1}{C^2} \sum_{n=1}^{\infty} \frac{1}{n^{1+2\varepsilon}},$$

the required bound (89) with the given value of C is valid for all n. The σ-measure of the set of values of λ, on which (89) is satisfied at least for some value $C = C(\lambda; \varepsilon)$, is therefore equal to

$$\lim_{C \to \infty} \left(1 - \frac{1}{C^2} \sum_{n=1}^{\infty} \frac{1}{n^{1+2\varepsilon}} \right) = 1,$$

Q. E. D.

Relation (84) was used in the proof only for $m = n$. Thus, the fact that $\{P_n(\lambda)\}$ is a sequence of polynomials and that they are mutually orthogonal was not used. This fact gives some grounds for the assumption that Theorem 9 and its continual analogue (Theorem 6 of No. 55) may be extended.

It is natural to associate the problems considered in Nos. 53 and 55 with the continual problem of V. A. Steklov, which, compared with its discrete analog, is little studied. We note that the verification of the continual analogue of V. A. Steklov's hypothesis is connected with the justification of the scheme usually used in the study of the energy spectrum of the one-dimensional Schroedinger operator L in the physical literature. The trouble is that papers and textbooks on quantum mechanics depart from the definition which follows from the postulates of quantum mechanics and which is in full agreement with the spectral theory of operators in Hilbert space. It is usually considered there that the spectrum $S(L)$ is the set of values of λ for which the solution $\varphi(x, \lambda)$ of equation (31) remains bounded for $x \to \infty$.

Although this assumption is not proved in the general case, in each concrete problem this approach is justified ultimately by obtaining expansions in series or integrals of special functions, for which the inversion formulas have already been studied within the framework of classical analysis. It is obvious that such a situation, when the solution of the problem is not reducible to spectral expansion, cannot satisfy the requirements of qualitative spectral analysis.

Thus, the problem arises of reconciling the "official" and "working" definitions of the spectrum.

Theorem 6 of No. 55, due to E. E. Shnol', is the first result related to this problem. It is impossible to improve this result by replacing the whole right-hand side of (41) of No. 55 by $C(\lambda)$ (see /52/); since, as was shown by V. P. Maslov /68/, without further assumptions regarding the spectral function $\rho(\lambda)$ the statement that $\varphi(x, \lambda)$ is bounded almost everywhere with respect to the ρ-measure is incorrect. However the corresponding example, constructed by V. P. Maslov /68/, does not contradict the continuum analogue of the hypothesis of V. A. Steklov, since for this example no inequality of the type (86) was established. We note that V. P. Maslov /68/ has succeeded in obtaining a continual analogue of the estimate (87) Some further results related to the problem of V. A. Steklov for polynomials were recently obtained by Ya. L. Geronimus /30 (3)/. However it is an open question, both in the discrete and in the continuous case, whether V. A. Steklov's hypothesis is correct.

Chapter VI

SOME ADDITIONAL PROBLEMS OF QUALITATIVE SPECTRAL ANALYSIS

59. Various generalizations of A. M. Molchanov's criterion. It was pointed out in No. 47 that the one-dimensional discreteness criterion, established in No. 28, is considerably complicated when passing to the multi-dimensional operator

$$-\Delta u + q(x)u \tag{1}$$

in $\mathscr{L}_2(\mathfrak{E}_m)$. In this connection it is interesting to note that for the more complicated case of the operator

$$(-1)^n \Delta^n u + q(x)u. \tag{2}$$

in contrast with the operator (1), A. M. Molchanov's one-dimensional criterion is unchanged, provided the order of the differential operator (2) is higher than the dimension of the space ($2n > m$). This was observed by M. Sh. Birman and B. S. Pavlov /14/. Their proof is based on an inequality similar to inequality (17) of No. 47 used in the one-dimensional case. It is of the form

$$\int_R |u^2(x) - u^2(x_0)|\,dx = \varepsilon(\omega) \int_R \left(\sum |D^{(n)}u|^2 + u^2\right) dx, \tag{3}$$

where $2n > m$, R is an m-dimensional cube with side ω, x_0 is any point of Ω, $\sum |D^{(n)}u|^2$ is the sum of squares of all the derivatives of order n of the function u, and

$$\lim_{\omega \to 0} \varepsilon(\omega) = 0.$$

Inequality (3) was obtained in /14/ by means of integral representations of functions in terms of their derivatives, similar to the well-known representations of S. L. Sobolev (see /90(2)/ and /89/).

Thus, in the case when the potential $q(x)$ is bounded from below, a necessary and sufficient condition for the spectrum of the operator (2) with $2n > m$ ($m \geqslant 1$) to be discrete is that

$$\int_R q(x)\,dx \to \infty. \tag{4}$$

when the cube R is displaced to infinity while preserving its dimension.

Other generalizations of A. M. Molchanov's criterion are connected with freeing it of the requirement that the potential be bounded from below Indeed, since we look for a condition for the boundedness from below and the discreteness of the spectrum, the only obligatory requirement is the

semiboundedness of the operator (1) itself. However, no one has succeeded in generalizing A. M. Molchanov's criterion that far.

In describing existing generalizations, we confine ourselves to the one-dimensional Schroedinger operator

$$-y''+q(x)y \qquad (-\infty<x<\infty). \tag{5}$$

The first generalization is due to I. Brink /15/, who replaced the requirement that the function $q(x)$ be bounded from below by the requirement that for certain positive γ and δ,

$$\int_t^x q(s)\,ds > -\gamma \tag{6}$$

in the domain $0<x-t<\delta$. A. M. Molchanov's discreteness criterion remains valid in the form (4). It is obvious that I. Brink's theorem permits the inclusion of potentials, unbounded from below, but rapidly oscillating.

However, as was recently observed by R. S. Ismagilov, the form (4) of A. M. Molchanov's criterion for the discreteness of the spectrum of the operator (5) is not universal. In his note /46/ he gives an example of an operator (5) whose spectrum is discrete and bounded from below and for which, despite (4), we have for all integer n

$$\int_n^{n+1} q(x)\,dx = 0.$$

Weakening condition (6) and using the localization principles (see No. 11), R. S. Ismagilov obtained a new form of the criterion for the discreteness of the spectrum. His results reduce the following /46/:

If there exist two measurable functions $\alpha(x)$ and $\beta(x)$ such that for some δ we have in the strip $0<x-t<\delta$

$$\int_t^x q(s)\,ds > \alpha(x)+\beta(t) \tag{7}$$

and for some $C<\infty$

$$\int_\omega \{\alpha^2(x)+\beta^2(x)\}\,dx \leqslant C, \tag{8}$$

for all the segments ω of unit length, then the operator (5) is semi-bounded. In particular, this holds in the case of condition (6) of I. Brink, since in (7) it is possible to take $\alpha(x)\equiv-\gamma$, $\beta(x)\equiv 0$, and then in (8) we have $C=\gamma^2$.

If the potential $q(x)$ satisfies conditions (7) and (8), then the following condition is necessary and sufficient for the spectrum of the operator (5) to be discrete: if the square R_a of the plane (x, t), determined by the inequalities

$$0 \leqslant x-t \leqslant \delta, \quad a \leqslant x+t \leqslant a+\delta, \tag{9}$$

goes to infinity, the function of the two variables

$$Q(x, t) = \int_t^x q(s)\,ds,$$

considered on the squares R_a, tends to infinity with respect to measure.

This means that for any M the measure of the set $\mathfrak{M}_a \subset R_a$ on which $Q(x, t) < M$ tends to zero for $a \to \infty$.

We also mention V. B. Lidskii's generalization (see /63 (2)/, /63 (3)/) of A. M. Molchanov's criterion to complex-valued potentials

$$q(x) = q_1(x) + iq_2(x).$$

This generalization states that in the case when the function $q_1(x)$ is bounded from below and the function $q_2(x)$ is semi bounded, the criterion for the discreteness of the spectrum of the operator (5) has the form (4), where the integrand $q(\mathbf{x})$ should be replaced by $q_1(x) + |q_2(x)|$.

In conclusion, we note that there is no known generalization of the discreteness criterion to matrix potentials.

60. The negative part of the spectrum of multidimensional two-term higher-order operators. As in the previous subsection, results of simple form on the negative part of the spectrum of the operator (2) are obtained for $2n > m$. The corresponding inclusion theorems led M. Sh. Birman and B. S. Pavlov /14/ to the following conclusions:

A criterion for the h-stable discreteness of the negative part of the spectrum of the operator (2) for $2n > m$ and $q(\mathbf{x}) \leqslant 0$ is that

$$\lim_{|\mathbf{x}| \to \infty} \int_{|\mathbf{z}-\mathbf{x}| < a} q(\mathbf{z})\, d\mathbf{z} = 0. \tag{10}$$

This relation also ensures for any sign of the potential that the continuous part of the spectrum coincides with the semi-axis $\lambda \geqslant 0$.

A criterion for the h-stable finiteness of the negative part of the spectrum of the operator (2) for $2n > m$ and $q(\mathbf{x}) \leqslant 0$ is that

$$\lim_{b \to \infty} b^{2n-m} \int_{|\mathbf{x}| > b} q(\mathbf{x})\, d\mathbf{x} = 0$$

or

$$\lim_{b \to \infty} \ln b \int_{|\mathbf{x}| > b} |\mathbf{x}|^{2n-m} q(\mathbf{x})\, d\mathbf{x} = 0$$

for odd and even m respectively.

In the one-dimensional case, the above results coincide with those established in Chapter II.

For $2n \leqslant m$, as in the statement of the theorem of A. M. Molchanov for the multidimensional Schroedinger operator, certain difficulties arise.

However, criterion (10) also holds for $2n \leqslant m$, if we assume in addition that the potential is bounded. For $m \geqslant 3$ and $n = 1$, a sufficient condition for the h-stable finiteness of the negative part of the spectrum of the operator (2) is given by

$$\lim_{\rho \to \infty} \rho \int_{\rho}^{\infty} q^*(r, \theta)\, dr = 0, \tag{11}$$

where θ means the set of all angular coordinates. As was shown in No. 30, in the case of a negative potential and $m = 1$ condition (11) is also necessary. This condition remains necessary for any $m \geqslant 3$ for potentials $q(\mathbf{x}) = q(|\mathbf{x}|) \leqslant 0$. For $m = 2$ condition (11) is replaced by

$$\lim_{\rho \to \infty} \ln \rho \int_{\rho}^{\infty} r q^*(r, \theta)\, dr = 0.$$

61. Algebraic investigation of the spectrum of the energy operator of a harmonic oscillator. The form of the stationary Schroedinger equation for the one-dimensional harmonic oscillator is

$$-y''+x^2y=\lambda y \qquad (-\infty<x<\infty). \tag{12}$$

The operator $\mathscr{H}$ defined by the left-hand side of this equation expresses the total energy of the oscillator.

Setting, in $\mathscr{L}_2(-\infty, \infty)$, $\mathscr{P}=I\frac{d}{dx}$, and denoting by G the operator of multiplication by x, the operator $\mathscr{H}$ may be represented in the form

$$\mathscr{H}=\mathscr{P}^2+G^2, \tag{13}$$

where

$$\mathscr{P}G-G\mathscr{P}=iI. \tag{14}$$

By means of relations (13) and (14) only, Dirac /41/ found the spectrum of the self-adjoint operator $\mathscr{H}$ in the abstract Hilbert space H. The account given below is due basically to N. I. Akhiezer.

Since

$$(\mathscr{P}+iG)(\mathscr{P}-iG)=\mathscr{P}^2-i(\mathscr{P}G-G\mathscr{P})+G^2,$$

then

$$(\mathscr{P}+iG)(\mathscr{P}-iG)=\mathscr{H}-I \tag{15}$$

and similarly,

$$(\mathscr{P}-iG)(\mathscr{P}+iG)=\mathscr{H}+I. \tag{16}$$

Multiplying relation (15) to the left and (16) to the right by $\mathscr{P}-iG$, we obtain the relation

$$(\mathscr{P}-iG)(\mathscr{H}-I)=(\mathscr{H}+I)(\mathscr{P}-iG) \tag{17}$$

and, similarly, the relation,

$$(\mathscr{H}-I)(\mathscr{P}+iG)=(\mathscr{P}+iG)(\mathscr{H}+I). \tag{18}$$

From relation (15) it follows that the operator $\mathscr{H}-I$ is nonnegative, so that the required spectrum $S(\mathscr{H})$ lies in the interval $[1, \infty]$.

Let us now assume that the operator $\mathscr{H}$ has at least one eigenvalue $\lambda \geqslant 1$, so that

$$\mathscr{H}f=\lambda f \qquad (\|f\|=1), \tag{19}$$

and set

$$g=(\mathscr{P}-iG)f.$$

If $g=0$, then by virtue of (15) we have

$$\mathscr{H}f=f,$$

so that $\lambda=1$.

If $g\neq 0$, then by virtue of (17) we have

$$(\lambda-1)g=(\mathscr{H}+I)g,$$

hence

$$\mathscr{H}g=(\lambda-2)g.$$

Thus the number $\lambda-2$ is also an eigenvalue.

Repeating the previous arguments, we find that either $\lambda-2=1$, i.e., $\lambda=3$, or the number $\lambda-4$ also is an eigenvalue. Since the operator $\mathscr{H}$ is bounded from below, after further repetition of this reasoning the second case ultimately becomes impossible. Thus, the number $\alpha=1$ is an eigenvalue of the operator $\mathscr{H}$, and its initial eigenvalue λ is an odd integer.

On the other hand, it follows from formula (18) that if (19) holds, the element

$$g=(\mathscr{P}+i\mathscr{G})f$$

satisfies the equation

$$\mathscr{H}g=(\lambda+2)g.$$

In this case $g\neq 0$, since otherwise it would follow from (16) that $\mathscr{H}f+f=0$, contradicting the nonnegativeness of the operator $\mathscr{H}-I$.

The above arguments establish that only positive odd numbers may be eigenvalues of the operator $\mathscr{H}$, and they are all eigenvalues of $D(\mathscr{H})\neq 0$.

Theorem 1. The spectrum of the self-adjoint operator $\mathscr{H}$ consists of the eigenvalues

$$\lambda_k=2k+1 \qquad (k=0,\ 1,\ \ldots). \tag{20}$$

To prove the theorem it remains to show that no number μ, differing from an odd number, may belong to $C(\mathscr{H})$.

Let E_λ denote the spectral function of the operator $\mathscr{H}$. We assume that $\mu\bar{\in}D(\mathscr{H})$ but $\mu\in C(\mathscr{H})$. Then for sufficiently small $\varepsilon>0$ a normalized element f may be found in $E(\Delta)H$ for $\Delta=(\mu-\varepsilon,\ \mu+\varepsilon)$, which is different from an eigenvector of the operator $\mathscr{H}$. For the element f we have

$$\|\mathscr{H}f-\mu f\|<\varepsilon.$$

From (15) the element

$$g=(\mathscr{P}-i\mathscr{G})f$$

differs from zero, and from (17) we have

$$(\mathscr{P}-i\mathscr{G})(\mathscr{H}-I)f=(\mathscr{H}+I)g$$

or

$$(P-i\mathscr{G})(\mathscr{H}f-\mu f)+(\mu-1)g=(\mathscr{H}+I)g.$$

that is

$$\mathscr{H}g-(\mu-2)g=(\mathscr{P}-i\mathscr{G})f_1. \tag{21}$$

where

$$f_1=\mathscr{H}f-\mu f.$$

Since the operator $\mathscr{H}$ is bounded in $E(\Delta)H$, and $f_1\in E(\Delta)H$, the quantity

$$\|(\mathscr{P}-i\mathscr{G})f_1\|^2=(|\mathscr{H}-I|f_1,\ f_1)$$

may be made as small as we please together with ε.

Thus, the norm of the right-hand side of (21) may with an appropriate choice of the element g become arbitrarily small. Hence it follows that the number $(\mu-2)$ belongs to $C(\mathscr{H})$. Repeating these arguments, we ultimately obtain that some negative number belongs to $C(\mathscr{H})$, and this contradicts the nonnegativeness of the operator $\mathscr{H}-I$.

The set $C(\mathscr{H})$ is therefore empty, and the theorem is proved.

Relations (13) and (14) do not imply the simplicity of the spectrum of the operator $\mathscr{H}$. However, it is easy to establish that all the eigenvalues of the singular problem (12) are simple. Thus, for example, the simplicity of the first eigenvalue follows from the fact that its corresponding eigenfunction should not have zeros, and then the simplicity of the remaining eigenvalues follows from the above considerations.

If the spectrum of the operator $\mathscr{H}$ is simple, and $\{e_k\}_{k=0}^{\infty}$ is the orthonormalized base of the eigenvectors corresponding to the spectrum (20), then by virtue of the argument preceding the formulation of the theorem, we have

$$\left.\begin{aligned}(\mathscr{P}-l\mathscr{G})\,e_0&=0,\\(\mathscr{P}-l\mathscr{G})\,e_k&=\alpha_k e_{k-1},\\(\mathscr{P}+l\mathscr{G})\,e_k&=\beta_k e_{k+1}\end{aligned}\right\}\quad (k=1,\ 2,\ \ldots). \tag{22}$$

Furthermore,

$$\begin{aligned}\alpha_k(e_{k-1},\ e_{k-1})&=([\mathscr{P}-i\mathscr{G}]\,e_k,\ e_{k-1})=\\&=(e_k,\ [\mathscr{P}+l\mathscr{G}]\,e_{k-1})=\overline{\beta}_{k-1}(e_k,\ e_k),\end{aligned}$$

so that

$$\alpha_k=\overline{\beta}_{k-1}\qquad (k=1,\ 2,\ \ldots).$$

On the other hand,

$$\begin{aligned}|\alpha_k|^2(e_{k-1},\ e_{k-1})&=([\mathscr{P}-l\mathscr{G}]\,e_k,\ [\mathscr{P}-l\mathscr{G}]\,e_k)=\\&=([\mathscr{H}-l]\,e_k,\ e_k)=2k\qquad (k=1,\ 2,\ \ldots).\end{aligned}$$

Without loss of generality, the numbers $l\alpha_k$ may be assumed positive. Indeed, if $l\alpha_k=|\alpha_k|e^{l\varphi_k}$, after passing to the base

$$e_k'=e_ke^{-i\psi_k}$$

we would obtain

$$(\mathscr{P}-l\mathscr{G})\,e_k'=-e^{l(\varphi_k-\psi_k+\psi_{k-1})}|\alpha_k|\,e_{k-1}'.$$

It would now be sufficient to determine the numbers ψ_r from the conditions

$$\psi_0=0,\quad \psi_r-\psi_{r-1}=\varphi_r\qquad (r=1,\ 2,\ \ldots).$$

Thus, it is possible to take in (22)

$$\alpha_k=-i\sqrt{2k}\qquad \beta_{k-1}=i\sqrt{2k},$$

leading to

$$\begin{aligned}\mathscr{G}e_k&=\frac{1}{2i}\,[(\mathscr{P}+l\mathscr{G})\,e_k-(\mathscr{P}-i\mathscr{G})\,e_k]=\\&=\frac{1}{2i}\,(\beta_ke_{k+1}-\alpha_ke_{k-1})=\sqrt{\frac{k+1}{2}}\,e_{k+1}+\sqrt{\frac{k}{2}}\,e_{k-1},\\\mathscr{P}e_k&=\frac{1}{2}\,[(\mathscr{P}-i\mathscr{G})\,e_k+(\mathscr{P}+l\mathscr{G})\,e_k]=\\&=\frac{1}{2i}\,(\alpha_ke_{k-1}+\beta_ke_{k+1})=i\sqrt{\frac{k+1}{2}}\,e_{k+1}-i\sqrt{\frac{k}{2}}\,e_{k-1}.\end{aligned}$$

Thus, in the base $\{e_k\}$, the operator $\mathscr{H}$ is represented by the diagonal matrix with the elements $h_{kk}=2k+1$, and the operators $\mathscr{P}$ and $\mathscr{G}$ are

represented by Jacobian matrices with the elements

$$p_{k,k+1}=\bar{p}_{k+1,k}=-i\sqrt{\frac{k}{2}}, \quad p_{kk}=0$$

and

$$q_{k,k+1}=q_{k+1,k}=\sqrt{\frac{k}{2}}, \quad q_{kk}=0$$

respectively.

The solution of the problem on the basis of relations (13) and (14) only, without the additional assumption that the spectrum is simple, was given by F. Rellich /83 (3)/.

62. The Dirac operator. This operator in the space $\mathscr{L}_2(\mathfrak{E}_3)$ of the four-component vector-functions $\varphi(\mathbf{x})=\{\varphi_k(\mathbf{x})\}_{k=1}^4$ has the form

$$T\varphi=(\alpha_1 p_1+\alpha_2 p_2+\alpha_3 p_3+\alpha_4)\varphi+r(\mathbf{x})\varphi,$$

where $p_k=-i\frac{\partial}{\partial x_k}$. $r(\mathbf{x})$ is the scalar potential of the external field, and α_k is four numerical fourth-order Hermitian matrices, satisfying the conditions

$$\alpha_j\alpha_k=-\alpha_k\alpha_j \qquad (j\neq k). \tag{23}$$

the square of each matrix being equal to the unit matrix E.

In the case of a free relativistic electron $r(\mathbf{x})=0$, so that the Dirac operator takes the form

$$T_0\varphi=P\varphi+\alpha_4\varphi.$$

where

$$P\varphi=\alpha_1 p_1+\alpha_2 p_2+\alpha_3 p_3.$$

To determine the spectrum $S(T_0)$ we start from the easily verifiable algebraic relation

$$T_0^2=P^2+I. \tag{24}$$

Since the spectrum of the operator P obviously covers the whole axis, then from (24) the spectrum $S(T_0^2)$ covers the semi-axis $[1,\infty)$. Therefore, at least one of any two real values of λ having equal modulus and satisfying the inequality $|\lambda|\geqslant 1$ belongs to $S(T_0)$. To establish that both these values belong to the spectrum $S(T_0)$ it is sufficient to show the symmetry of the set $S(T_0)$ with respect to the point $\lambda=0$.

For this purpose, we make use of the fifth Hermitian matrix, $\alpha_5(\alpha_5^2=E)$, satisfying relations (23) together with the matrices α_k $(k=1,2,3,4)$ (see /96/).

Now, if $\lambda\in S(T_0)$, for any $\varepsilon>0$ we have for some vector-function φ

$$\|T_0\varphi-\lambda\varphi\|<\varepsilon\|\varphi\|.$$

But then for the vector-function $\alpha_5\varphi$

$$T_0\alpha_5\varphi+\lambda\alpha_5\varphi=-\alpha_5(T_0\varphi-\lambda\varphi),$$

so that

$$\|T_0\alpha_5\varphi+\lambda\alpha_5\varphi\|=\|T_0\varphi-\lambda\varphi\|<\varepsilon\|\varphi\|=\varepsilon\|\alpha_5\varphi\|$$

consequently, $(-1)\lambda\in S(T_0)$.

Thus, the spectrum $S(T_0)$ covers the whole axis, with the exception of the interval gap $(-1,1)$.

We now consider the operator T as the perturbation of the operator T_0 by the potential $r(\mathbf{x})$ and study the spectrum introduced by this perturbation in the gap $(-1, 1)$. This problem is obviously equivalent to the study of the negative part of the spectrum of the operator $T^2 - I$. In order to obtain the conditions for the discreteness and the finiteness of the spectrum of the operator T in the interval $(-1, 1)$, we consider the operator $T^2 - \beta^2 I$, where $0 < \beta \leqslant 1$.

We have

$$\begin{aligned}(T^2\varphi, \varphi) &= (|T_0 + r|^2\varphi, \varphi) = \\ &= (T_0^2\varphi, \varphi) + 2\,\mathrm{Re}\,(T_0\varphi, r\varphi) + (r^2\varphi, \varphi) = \\ &= (P^2\varphi, \varphi) + (\varphi, \varphi) + 2\,\mathrm{Re}\,(P\varphi, r\varphi) + \\ &\qquad + 2\,\mathrm{Re}\,(\alpha_4\varphi, r\varphi) + (r^2\varphi, \varphi),\end{aligned}$$

so that

$$\begin{aligned}(|T^2 - \beta^2 I|\varphi, \varphi) &= \\ &= (|P + r|^2\varphi, \varphi) + 2\,\mathrm{Re}\,(\alpha_4\varphi, r\varphi) + (1 - \beta^2)(\varphi, \varphi).\end{aligned}$$

If

$$(|T^2 - \beta^2 I|\varphi, \varphi) < 0, \tag{25}$$

then a fortiori

$$(|P + r|^2\varphi, \varphi) - 2(\varphi, |r|\varphi) + (1 - \beta^2)(\varphi, \varphi) < 0$$

and from the elementary inequality

$$\|(P + r)\varphi\|^2 > \tfrac{1}{2}\|P\varphi\|^2 - \|r\varphi\|^2,$$

we have

$$(P^2\varphi, \varphi) - 2((r^2 + 2|r|)\varphi, \varphi) + \varepsilon(\varphi, \varphi) < 0,$$

where $\varepsilon = 2(1 - \beta^2)$.

The latter inequality may be represented in the form

$$\int_{\xi} \{|\nabla\varphi|^2 - 2(r^2 + 2|r|)|\varphi|^2\}\,d\mathbf{x} < -\varepsilon \int_{\xi} |\varphi|^2\,d\mathbf{x}, \tag{26}$$

where

$$|\varphi|^2 = \sum_{k=1}^{4} |\varphi_k|^2, \quad |\nabla\varphi|^2 = \sum_{k=1}^{4} |\nabla\varphi_k|^2.$$

The integral on the left-hand side of inequality (26) is a quadratic functional, generated in $\mathfrak{L}_2(\xi)$ by the Schroedinger operator

$$-\Delta\varphi - 2(2|r| + r^2)\varphi. \tag{27}$$

This enables us to use the theorems on the negative part of the spectrum of the Schroedinger operator.

If the part of the spectrum of the operator (27) lying to the left of the point $-\varepsilon < 0$ is exhausted by a finite set of eigenvalues of finite multiplicity, the maximum dimension of the linear manifolds on which inequality (26) is satisfied is finite. Consequently, the maximum dimension of the manifold on which inequality (25) is satisfied is also finite. This means that the part of the spectrum of the operator T^2 lying to the left of the point β^2 is exhausted

by a finite set of eigenvalues of finite multiplicity. Consequently, the part of the spectrum of the operator T lying in the interval $(-\beta, \beta)$ is exhausted by a finite set of eigenvalues of finite multiplicity.

Thus, if the potential $-2(2|r|+r^2)$ of the Schroedinger operator (27) satisfies some condition for the discreteness or the finiteness of the negative part of the spectrum (see Nos. 46 and 48), the perturbation $r(\mathbf{x})$ introduces in the gap $(-1, 1)$ of the spectrum of the Dirac operator T_0 only an isolated (respectively, a finite) set of eigenvalues.

Concrete examples of conditions for the discreteness or the finiteness of the spectrum $S(T)$ in the interval $(-1, 1)$ are given by Theorems 2 and 6 of Chapter IV.

A rigorous investigation of the spectrum $S(T)$ in the interval $(-1, 1)$ when the potential $r(\mathbf{x})$ is bounded was carried out by M. Sh. Birman in his paper /13 (7)/, where the following theorem is proved.

Theorem 2. /13 (7)/. If a free relativistic electron is perturbed by a bounded potential $r(\mathbf{x})$ whose absolute value $|r(\mathbf{x})|$ satisfies a condition of the form (10), then $C(T)=C(T_0)$ so that in the gap $(-1,1)$ of the spectrum $S(T_0)$ only a discrete spectrum may appear. If, in addition, $r(\mathbf{x})\geqslant 0$ for large $|\mathbf{x}|$ this discrete spectrum cannot admit the point $\lambda=+1$ as a cluster point. Finally, if the potential $r(\mathbf{x})$ satisfies a condition of the form (11), then in the interval $(-1,1)$ also the operator T has only a finite number of eigenvalues.

In /13 (7)/ upper bounds for the number of eigenvalues of the operator T in the interval $(-\beta, \beta)$ for $|\beta|<1$ were also obtained.

Sufficient criteria for the infinity of the set of eigenvalues of the operator T contained in the gap $(-1, 1)$ under condition (10) are not known.

63. The Pauli operator. The Pauli wave equations for an electron in a magnetic field are of the form /96/

$$P\psi_1 - eq\psi_1 + \mu_0\{H_z\psi_1 + (H_x - iH_y)\psi_2\} = i\hbar\frac{\partial\psi_1}{\partial t},$$

$$P\psi_2 - eq\psi_2 + \mu_0\{(H_x + iH_y)\psi_1 + H_z\psi_2\} = i\hbar\frac{\partial\psi_2}{\partial t},$$

where

$$P\psi_k = -\frac{\hbar^2}{2m}\Delta\psi_k - \frac{i\hbar e}{mc}\langle \boldsymbol{A}, \nabla\psi_k\rangle + \frac{e^2}{2mc^2}|\boldsymbol{A}|^2\psi_k \quad (k=1, 2).$$

Here given functions of the point $\mathbf{x}\in\mathfrak{E}$ are: the vector $\boldsymbol{A}$ (the vector potential of the magnetic field), satisfying the condition $\operatorname{div}\boldsymbol{A}=0$; the vector $\boldsymbol{H}=\operatorname{rot}\boldsymbol{A}$ (the magnetic field intensity), and the scalar q (the potential energy). The triangular brackets denote the scalar product in $\mathfrak{E}$.

The Pauli operator thus has the form

$$P\psi + Q(\mathbf{x})\psi,$$

where $\psi=\{\psi_1, \psi_2\}$ is a two-component vector-function of the point $P\psi=\{P\psi_1, P\psi_2\}$ and

$$Q(\mathbf{x}) = \mu_0\begin{pmatrix} H_z - \frac{e}{\mu_0}q & H_x + iH_y \\ H_x - iH_y & -H_z - \frac{e}{\mu_0}q \end{pmatrix}$$

The largest and the smallest eigenvalues of the matrix $Q(\mathbf{x})$ are

respectively equal to

$$\nu_1(\mathbf{x}) = -eq(\mathbf{x}) - \mu_0 H(\mathbf{x}), \quad \nu_2(\mathbf{x}) = -eq(\mathbf{x}) + \mu_0 H(\mathbf{x}),$$

where

$$H(\mathbf{x}) = \sqrt{H_x^2 + H_y^2 + H_z^2}.$$

A smooth real finite vector-function satisfies the identity

$$(P\psi, \psi) = \sum_{k=1}^{2} (P\psi_k, \psi_k) =$$

$$= \sum_{k=1}^{2} \int_{\mathfrak{G}} \left\{ -\frac{\hbar}{2m} \psi_k \Delta\psi_k - \frac{ihe}{mc} \langle A\psi_k, \nabla\psi_k \rangle + \frac{e^2}{2mc^2} |A|^2 \psi_k^2 \right\} d\mathbf{x} =$$

$$= \sum_{k=1}^{2} \int_{\mathfrak{G}} \left\{ \frac{\hbar^2}{2m} |\nabla\psi_k|^2 - \frac{ihe}{mc} \langle A\psi_k, \nabla\psi_k \rangle + \frac{e^2}{2mc^2} |A|^2 \psi_k^2 \right\} d\mathbf{x} =$$

$$= \frac{1}{2m} \sum_{k=1}^{2} \int_{\mathfrak{G}} \left| ih\nabla\psi_k - \frac{e}{c} A\psi_k \right|^2 d\mathbf{x}$$

from which it follows that the operator P is nonnegative in $\mathfrak{L}_2(\mathfrak{G})$.

From No. 36 we obtain that when the condition

$$\lim_{|x| \to \infty} \{eq(\mathbf{x}) + \mu_0 H(\mathbf{x})\} = -\alpha$$

is satisfied, the part of the spectrum of the Pauli operator (in $\mathfrak{L}_2(\mathfrak{G})$) lying to the left of the point α is bounded from below and discrete. Also, if the condition

$$\lim_{|x| \to \infty} \{eq(\mathbf{x}) + \mu_0 H(\mathbf{x})\} = -\infty$$

is satisfied, the whole spectrum of the Pauli operator is discrete with the single limit point $\lambda = +\infty$ /31(7)/.

Suppose

$$\lim_{|x| \to \infty} q(\mathbf{x}) = \lim_{|x| \to \infty} H(\mathbf{x}) = 0. \tag{28}$$

In addition we assume that for a certain $\delta > 0$ and $|\mathbf{x}| \to \infty$

$$H(\mathbf{x}) = o(|\mathbf{x}|^{-(1+\delta)}). \tag{29}$$

With these assumptions we now generalize over the whole $\mathfrak{G}$ a formula known for bounded domains /106/ obtaining

$$A = \frac{1}{4\pi} \int_{\mathfrak{G}} \frac{[e(\mathbf{x}, z) \times H(z)]}{|\mathbf{x} - z|^2} dz, \tag{30}$$

where $e(\mathbf{x}, z)$ is the unit vector in the direction $\mathbf{x} - z$.

From (29) and (30) we obtain

$$\lim_{|x| \to \infty} A(\mathbf{x}) = 0. \tag{31}$$

We show that from this relation it follows that the positive part of the spectrum of the operator

$$-\Delta u + a\langle A, \nabla u \rangle + b|A|^2 u \tag{32}$$

is continuous. Here a and b are constant coefficients.

Since the positive part of the spectrum of the operator $-\Delta$ covers the semi-axis $\lambda \geqslant 0$, then for $\lambda_0 \geqslant 0$ there exists a system of smooth normalized

finite functions $\{u_n(\mathbf{x})\}_{n=1}^{\infty}$ with nonintersecting supports ω_n for which

$$\lim_{n\to\infty} \|\Delta u_n + \lambda_0 u_n\| = 0. \tag{33}$$

Moreover,

$$\langle \boldsymbol{A}, \nabla u_n\rangle\|^2 \leqslant \|(|\boldsymbol{A}| \cdot |\nabla u_n|)\|^2 \leqslant \max_{\mathbf{x}\in\omega_n} |\boldsymbol{A}(\mathbf{x})|^2 \int_{\omega_n} |\nabla u_n|^2\, d\mathbf{x} =$$

$$= \max_{\mathbf{x}\in\omega_n} |\boldsymbol{A}(\mathbf{x})|^2 \int_{\omega_n} |\Delta u_n \cdot u_n|\, d\mathbf{x} \leqslant \max_{x\in\omega_n} |\boldsymbol{A}(\mathbf{x})|^2 \|\Delta u_n\|.$$

Since from (33) the quantities $\|\Delta u_n\|$ are bounded for $n\to\infty$, from (31) and the latter inequality it follows that

$$\lim_{n\to\infty} \|\langle \boldsymbol{A}, \nabla u_n\rangle\| = 0.$$

Thus,

$$\lim_{n\to\infty} \| -\Delta u_n + a\langle \boldsymbol{A}, \nabla u_n\rangle + b|\boldsymbol{A}|^2 u_n - \lambda_0 u_n\| = 0,$$

from which it follows that the positive part of the spectrum of the operator (32) covers the semi-axis $\lambda \geqslant 0$.

Furthermore, since condition (28) implies that both the eigenvalues $\nu_1(\mathbf{x})$ and $\nu_2(\mathbf{x})$ of the matrix $Q(\mathbf{x})$ tend to zero for $|\mathbf{x}|\to\infty$, the norm of the matrix $Q(\mathbf{x})$ also tends to zero for $|\mathbf{x}|\to\infty$.

From No. 36 and the nonnegativeness of the operator P, we conclude that under conditions (28) and (29) the continuous part of the spectrum of the Pauli operator coincides with the semi-axis $\lambda \geqslant 0$ /31 (7)/.

64. The spectrum of the energy operator of a many-particle system. In the simplest case, when a stationary nucleus is taken as the origin of the coordinates and there are only two particles, the operator $\mathscr{H}$ has the form /34/

$$\mathscr{H}\psi \equiv -\Delta_1\psi - \Delta_2\psi + \{U_1(\mathbf{r}_1) + U_2(\mathbf{r}_2) + U_{12}(\mathbf{r}_1, \mathbf{r}_2)\}\psi, \tag{34}$$

where $\mathbf{r}_1(x_1, x_2, x_3)$ and $\mathbf{r}_2(x_4, x_5, x_6)$ are the radius-vectors of the particles,

$$\Delta_1 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}, \quad \Delta_2 = \frac{\partial^2}{\partial x_4^2} + \frac{\partial^2}{\partial x_5^2} + \frac{\partial^2}{\partial x_6^2},$$

and

$$U_{12}(\mathbf{r}_1, \mathbf{r}_2) = U_{12}(|\mathbf{r}_2 - \mathbf{r}_1|) \tag{35}$$

is the interaction potential of these particles, satisfying the condition

$$\lim_{r_{12}\to\infty} U_{12}(\mathbf{r}_1, \mathbf{r}_2) = 0, \tag{36}$$

where $r_{12} = |\mathbf{r}_{12}|$ and $\mathbf{r}_{12} = \mathbf{r}_2 - \mathbf{r}_1$.

Together with this condition we also assume that

$$\lim_{r_1\to\infty} U_1(\mathbf{r}_1) = \lim_{r_2\to\infty} U_2(\mathbf{r}_2) = 0. \tag{37}$$

The operator $\mathscr{H}$ can also be written in the form

$$\mathscr{H}\psi = -\Delta\psi + q(\mathbf{x})\psi, \tag{38}$$

where ψ is a function of the point $\mathbf{x}\in\mathfrak{E}_6$ and

$$q(\mathbf{x}) = U_1(\mathbf{r}_1) + U_2(\mathbf{r}_2) + U_{12}(\mathbf{r}_1, \mathbf{r}_2). \tag{39}$$

For the Coulomb interaction we obtain the operator

$$\mathscr{H}\psi = -\Delta\psi - \left(\frac{b_1}{r_1} + \frac{b_2}{r_2} - \frac{c_{12}}{r_{12}}\right)\psi, \tag{40}$$

which for certain values of the constants b_1, b_2 and c_{12} is the same as the energy operator of the helium atom.

Formally, the operator (38) is identical to that considered in Chapter IV. However, in the case of a system of particles there appear additional difficulties, connected with the fact that in general the continuously differentiable potential (39) becomes infinite not at individual points but over the manifolds $r_1 = 0$, $r_2 = 0$ and $r_1 = r_2$ that go to infinity.

We show that under conditions (36) and (37) there exists a number $\mu \leqslant 0$ such that all the values $\lambda \geqslant \mu$ belong to the continuous part of the spectrum of the operator $\mathscr{H}$ in $\mathfrak{L}_2(\mathfrak{E}_6)$. There is no need to assume that the function $U_{12}(r_1, r_2)$ is of the form (35).

To prove the existence of the number μ we choose in $\mathfrak{E}_6$ an arbitrary point $\mathbf{x}_0$ having positive and different coordinates. We draw from the origin of the coordinates through the point $\mathbf{x}_0$ a half-time O and denote by Ω the circular cone having O as its axis of rotation and whose angle at a apex is sufficiently small so that the coordinates of any point lying inside or on the boundary of Ω are positive and different.

If some point $\mathbf{x}$ goes to infinity while remaining inside the cone Ω, then each coordinate of the point $\mathbf{x}$ tends to infinity so that the function $q(\mathbf{x})$ defined by (39) satisfies the conditions

$$\lim_{|x|\to\infty} q(\mathbf{x}) = 0 \qquad (\mathbf{x} \in \Omega).$$

But then, by virtue of Theorem 8 of No. 49, we have $[0, \infty) \subset C(\mathscr{H})$, which implies the existence of the required number $\mu \leqslant 0$.

The operator (34) is a particular case of the operator

$$\mathscr{H}\psi = -\sum_{j,k=1}^{n}\sum_{\nu=0}^{2} a_{jk}\frac{\partial^2\psi}{\partial x_{3j-\nu}\partial x_{3k-\nu}} + \sum_{\substack{j,k=0\\(j<k)}} V_{jk}(r_{jk})\psi, \tag{41}$$

where $x_{3j-\nu}$ $(\nu = 0, 1, 2;\ j = 1, 2, \ldots, n)$ are the coordinates of a point in the $3n$-dimensional Euclidian space $\mathfrak{E}_{3n}$, $r_{jk} = r_j - r_k$,

$$r_j = (x_{3j-2}, x_{3j-1}, x_{3j}) \quad (j, k \geqslant 1), \quad r_{0j} = r_j \quad (j = 1, 2, \ldots, n)$$

and $a_{jk} = a_{kj}$ are the coefficients of a positive definite quadratic form. Neglecting the motion of the nucleus, $a_{jk} = 0$ for $j \neq k$. Formula (41) applies to a wide class of quantum-mechanical systems. Thus, for example, in the case of the Coulomb interaction for a system of any number of charged particles of the same sign situated in the field of a few stationary particles with charges of the opposite sign, we put in (41)

$$V_{0j}(r_{0j}) = -\sum_{s=1}^{n_0} b_{sj}\frac{1}{r_{a_s, j}} \qquad (j = 1, 2, \ldots, n), \tag{42}$$

$$V_{jk}(r_{jk}) = \frac{c_{jk}}{r_{jk}} \qquad (j, k = 1, 2, \ldots, n;\ j < k). \tag{43}$$

where n_0 is the number of the stationary particles, $\{a_{3s-\nu}\}$ $(\nu = 0, 1, 2;\ s = 1, 2, \ldots, n_0)$ are their coordinates, $r_{sj} = \left\{\sum_{\nu=0}^{2} (x_{3j-\nu} - a_{3s-\nu})^2\right\}^{1/2}$, and $c_{jk} = c_{kj}$ and b_{sj} are some nonnegative numbers.

The first mathematical investigation of the character of the energy spectrum of a system of two particles was undertaken by T. Kato /48 (1)/, who succeeded in showing that in the case of the helium atom, when in (40) $b_1 = b_2 = me^2\hbar^{-2}$, and $c_{12} = \frac{1}{2} me^2\hbar^2$, where m and e are the mass and charge of the electron, to the left of a certain point the spectrum of the operator $\mathcal{H}$ is discrete and contains a nonempty set of eigenvalues. The character of the continuous part of the spectrum remains unclarified. Following the method of T. Kato (described also in /95 (2)/), C. Putnam /79 (5)/ carried this result over to the case of strongly ionized atoms, when the order number of the atom in D. I. Mendeleev's table is greater than $\frac{5}{8} n(n-1)$ where n is the number of the remaining electrons.

The most complete results on the nature of the spectrum of the operator $\mathcal{H}$ are due to G. M. Zhislin. Developing the methods of the multidimensional singular variational caculus, Zhislin investigated the spectrum of the operator (41) in a number of papers /42/, where he made some specific assumptions. His general criterion for the existence of a discrete spectrum of the operator (41) (see /42 (4)/) is apparently the first proof (i.e., derivation from the basic equations of quantum mechanics) of an energy principle known in quantum chemistry as follows: in order that a quantum-mechanical system of n particles be stable, it is sufficient that the removal of any particle from the system be energetically unfavored /34/.

When applied to the Coulomb potential (42), (43), the results of Zhislin's results reduce to the following:

There always exists a number $\mu \leqslant 0$, such that $C(\mathcal{H}) = [\mu, \infty)$.

$$\sum_{s=1}^{n_0} b_{sk} > \sum_{j=1,\ j\neq k}^{n} c_{kj} \tag{44}$$

$(k = 1, 2, \ldots, n;\ n \geqslant 2)$, then $\mu < 0$, and to the left of the point μ the spectrum $S(\mathcal{H})$ is exhausted by an infinite increasing sequence of eigenvalues λ_m $(m = 1, 2, \ldots)$ which tends to μ. The multiplicity of the first eigenvalue is then equal to one. Condition (44) is satisfied for all known atoms, positive ions and molecules under the assumption that the nuclei of the atoms in the molecule are stationary.

65. The structure of the continuous part of the spectrum of the Schroedinger operator. In all the examples of quantum-mechanical problems with specific potentials, the structure of the continuous part of the spectrum of the Schroedinger operator L has turned out to be sufficiently simple. Usually, the continuous part of the spectrum does not contain any eigenvalues. When such eigenvalues exist they form an isolated set and the spectral function in the orthogonal complement to all the proper elements is always absolutely continuous. However, from the results of I. M. Gel'fand and B. M. Levitan on the theory of inverse problems of spectral analysis /28/ it follows directly that even in the one-dimensional case the continuous part of the spectrum may contain discrete and singular components of any complexity.

The problem of the eigenvalues in the continuous part of the spectrum is quite complicated. One may get an idea of the reasons for the complexity of this problem from the well-known theorem of Weyl-von Neumann, according to which it is possible by means of a completely continuous perturbation with an arbitrarily small absolute norm to transform the whole spectrum of a self-adjoint operator A into a purely point spectrum (see /89/). This means that even if prior to the perturbation the operator A had no eigenvalues the perturbed operator $\tilde{A}$ can have a complete system of proper elements in the Hilbert space H.

There are apparently no qualitative methods for discovering the eigenvalues in $C(L)$ from the behavior of the potential at infinity. It is somewhat simpler to show the absence of eigenvalues in $C(L)$. In this case, it is sufficient to show the absence of nontrivial solutions of the equations $Lu-\lambda u=0$ in $\mathfrak{L}_2$ for $\lambda\in C(L)$. In this manner all the known theorems on the absence of eigenvalues in $C(L)$ are proved.

In particular, T. Kato /48(3)/ has established that for $\lambda>0$ the solutions $u(x)$ of the equation $-\Delta u+qu=\lambda u$ cannot decrease too rapidly if

$$q(x)=o(|x|^{-1}). \tag{45}$$

More exactly, for any $a>0$, $b>0$ we have

$$\lim_{R\to\infty} R^a \int\limits_{R\leqslant|x|<R+b} |u(x)|^2\,dx=\infty.$$

Hence it follows in particular that the corresponding operator L in $\mathfrak{L}_2(\mathfrak{G})$, for which the continuous part of the spectrum coincides with the semi-axis $\lambda\geqslant 0$, cannot have positive eigenvalues under condition (45). This is true for the exterior of any bounded domain and for complex-valued potentials. The investigations of T Kato are based on complicated integro-differential inequalities; no simpler proof of the absence of points of $D(L)$ in $C(L)$ is known.

The abstract theorem of Weil-von Neumann characterizes the discrete part of the spectrum as its unstable component: when passing from A to B by means of a completely continuous perturbation K of arbitrarily small absolute norm, on the component $C(A)=C(B)$ of the spectrum of the operator A there may appear an everywhere dense set of eigenvalues, or conversely, a set of eigenvalues everywhere dense in $C(A)$ may entirely disappear. The discrete part of the spectrum that lies in $C(A)$ remains unstable also in the case of weaker perturbations and even in the case of one-dimensional perturbations. The extreme instability of this component and of the singular component of the spectrum of the one-dimensional Schroedinger operator was recently observed by N. Aronszajn. In his paper /4/ it is shown that for a given potential, these components may be changed one into the other by merely changing only one boundary condition at zero.

In contrast, the absolutely continuous part of the spectrum, which is not stable and which may even completely disappear under the conditions of the theorem of Weyl-von Neumann, is invariant under weaker perturbations. The corresponding result was obtained in 1957 by M. Rosenblum /85/ and T. Kato /48(2)/.

The Rosenblum-Kato theorem establishes that when a self-adjoint operator A is perturbed by a completely continuous operator with a finite absolute trace, the absolutely continuous part of the operator A is preserved

to within unitary equivalence. This theorem is an important generalization of H. Weyl's theorem on the preservation of the continuous part of the spectrum in the case of completely continuous perturbations. The Rosenblum-Kato theorem is closely related to the problem of the existence of scattering operators, introduced in quantum mechanics by W. Heisenberg.

When the Rosenblum-Kato theorem is used in the theory of singular differential operators, one has to pass from the operators themselves to their resolvents or to the powers of their resolvency.

A similar situation existed in the first applications of H. Weyl's theorem to the theory of singular differential operators. Thus, for example, in order to prove the invariance of the continuous part of the spectrum of the Schroedinger operator (1) outside a closed surface when this surface is deformed, A. Ya. Povzner has investigated in /8/ the difference in the resolvent kernels of the corresponding boundary-value problems.

Let us examine a problem dealing with the perturbation of the boundary and of the boundary conditions which was considered by M. Sh. Birman /13 (10)/ for the Schroedinger operator (1) on the basis of the Rosenblum-Kato theorem.

Let Ω be the exterior of the bounded domain ω in $\mathfrak{E}_m$, Γ be the common boundary of ω and Ω and let

$$\frac{\partial u}{\partial n} + \sigma_\Gamma(x)\, u\Big|_{x\in\Gamma} = 0 \qquad (\sigma_\Gamma(x) \geqslant 0). \tag{46}$$

We also assume that $q(x) \geqslant 0$ in (1). As was proved in No. 17, the set $C(L)$ does not change when the boundary conditions (46) are changed. Following M. Sh. Birman /13 (11)/, we consider a similar problem with respect to the absolutely continuous part of the spectrum $S(L_C) \subset C(L)$. For simplicity we assume that the first boundary condition $u|_{x\in\Gamma} = 0$ holds all the time on the changing boundary Γ, and denote by $\tilde{L}_\omega$ and $\tilde{L}_\Gamma$ the self-adjoint operators defined by this boundary condition and operation (1) in $\mathscr{L}_2(\omega)$ and $\mathscr{L}_2(\Omega)$, respectively. It is obvious that the operator $M = \tilde{L}_\omega \oplus \tilde{L}_\Gamma$, which is self-adjoint in $\mathscr{L}_2(\mathfrak{E}_m)$, is positive-definite and the singular properties of the spectrum of the operator M are generated by the operator $\tilde{L}_\Gamma$. It is therefore sufficient to compare the operator M with the operator L generated by operation (1) throughout the space $\mathscr{L}_2(\mathfrak{E}_m)$.

Let L_0 be the result of the decomposition of L on the surface γ. Then both operators L and M are positive-definite self-adjoint extensions of the operator L_0, M being the Fredrichs extension of L_0.

Furthermore, using the results of M. G. Krein /53 (2)/ and M. I. Vishik /24 (3)/ in the extension theory of semi-bounded operators, it can be shown that the eigenvalues of the positive operator $K = L^{-1} - M^{-1}$ do not exceed the successive maximums of the integral

$$\int_{\mathfrak{E}_m} |u(x)|^2\, dx$$

with the condition

$$\int_{\mathfrak{E}_m} (|\nabla u|^2 + q\,|u|^2)\, dx = 1$$

and the corresponding linear relations, where the function $u(x)$ satisfies the equation $-\Delta u + qu = 0$ inside and outside the domain ω. It may be shown

that these maximums λ_r are in fact attained, and the following estimate holds:

$$\lambda_r \leqslant Cr^{\frac{2}{1-m}}.$$

Hence for $m=2$ it immediately follows that the operator K has a finite absolute trace, and therefore the absolutely continuous parts of the operators L^{-1} and M^{-1} are unitarily equivalent.

For $m>2$, a similar result is obtained by considering the operator $L^{-p}-M^{-p}$ for a sufficiently large natural number p.

It is also easy to allow for the influence of the variation of a boundary condition of the form (46) itself. In the same way an operation (1) degenerating on parts of the boundary, and other similar problems, can be considered.

It should be noted, however, that in the scattering problem for operation (1) and in other closely related problems the use of the Rosenblum-Kato theorem has not yet given anything new when compared with the results obtained earlier by the method of A. Ya. Povzner.

The result of A. Ya. Povzner /81/, which was derived in 1953, and some improvements due to D. M. Eidus /110/ and T. Ikebe /45/, establish that for $m=3$ and $q(x)=O(|x|^{-(2+\delta)})(\delta>0)$ the absolutely continuous part of the spectrum coincides with the semi-axis $\lambda \geqslant 0$ and that there is no continuous singular spectrum. Closely related problems were studied by another method by O. A. Ladyzhensakya and L. D. Faddeev /48/, who continued the corresponding investigations of K. Friedrichs /97 (2)/. The method of these authors is insensitive to the dimension of the space.

We complete this short review of work done on the investigation of the structure of the continuous spectrum by noting that in this field sufficiently powerful general methods of qualitative analysis have not yet been developed. There are a number of concrete unsolved problems here. In particular, for the energy operator of the helium atom, the problem of the existence of a singular part of the spectrum and of the existence of eigenvalues in the continuous part of the spectrum remains open /40/.

66. Complex potentials. Such potentials first arose in quantum-mechanical problems with energy dissipation /8/. In general, the appearance of nonself-adjoint operators is characteristic of the description of closed physical systems. The basic investigations on the spectral theory of nonself-adjoint operators in the abstract Hilbert space were made by M. V. Keldysh /51 (2)/ and M. S. Lifshitz /62/. The first results on the spectral analysis of the one-dimensional Schroedinger operator with a complex potential were obtained by M. A. Naimark /72 (4)/ and B. Ya. Levin /60/.

This subsection is devoted to a review of some results dealing with the discrete part of the spectrum of the operators (1) and (5) with a complex potential. The methods and the results exhibit an essential difference with respect to the self-adjoint case, studied in Nos. 30 and 48.

Let L denote the minimum operator defined by operation (1) in $\mathfrak{L}_2(0, \infty)$. Assuming the defect number of the J-symmetric operator L equal to one, we denote by L_h the self-adjoint extension of the operator L, defined by the boundary condition

$$y(0)-hy'(0)=0 \qquad (47)$$

with the complex parameter h.

The requirements imposed in this subsection on the potential $q(x)$ imply that $C(L)$ coincides with the semi-axis $\lambda \geqslant 0$. Hence it follows that $\operatorname{Def} L = 1$, since if $\operatorname{Def} L = 2$ we would have $C(L) = 0$.

In the self-adjoint case if

$$\int_0^\infty x|q(x)|\,dx < \infty \tag{48}$$

the spectrum of the operator L_h consists of the positive λ-semi-axis and of a finite number of negative eigenvalues (see Theorems 23 of No. 31 and 10 of No. 30).

In the general case, when the potential $q(x)$ and the parameter h in the boundary condition (47) are complex, we also have $C(L_h) = [0, \infty)$. As for the part of the spectrum of the operator L_h situated outside the semi-axis $\lambda \geqslant 0$ it can only be said that it consists of a bounded set of eigenvalues of finite rank, all the limit points of which lie on the semi-axis $\lambda \geqslant 0$ (see /72(4)/ and /60/).

As a criterion for the finiteness of this set, M. A. Naimark /72(4)/ obtained instead of (48) the considerably more stringent condition

$$\int_0^\infty e^{\varepsilon x}|q(x)|\,dx < \infty, \tag{49}$$

where ε is some positive number. When this holds, the continuous part of the spectrum $C(L_h) = [0, \infty)$ does not contain eigenvalues.

Condition (49) guarantees, for $\operatorname{Im} z > -\varepsilon$ the analyticity of the function

$$\varphi(0, z) - h\varphi'_x(0, z), \tag{50}$$

where $\varphi(x, z)$ is the solution of the equation

$$-y'' + q(x)y = z^2 y,$$

belonging to $\mathfrak{L}_2(0, \infty)$ and satisfying the condition $\lim_{x\to\infty} e^{-izx}\varphi(x, z) = 1$. Even the weaker condition (48) yields that such a solution exists and is unique, for nonreal z /60/, and that the function (50) is analytic in the upper and lower z-half-planes and may be extended by continuity on the real axis. Since the nonreal roots of the function (50) exhaust the discrete part of the spectrum of the J-self-adjoint operator L_h lying outside the semi-axis $\lambda \geqslant 0$, the result of M. A. Naimark follows from the analyticity of this function.

However, B. Ya. Levin has drawn attention to the fact that for the finiteness of the discrete part of the spectrum of the operator L_h situated outside the semi-axis $\lambda \geqslant 0$ it is sufficient to require that the function (50) belong to some quasi-analytic class of functions on the real axis Accordingly, B. Ya. Levin /60/ has succeeded in somewhat weakening the requirement (49), replacing it by the condition

$$|q(x)| < e^{-\gamma(x)},$$

where the function $x\gamma'(x)$ tends monotonically to infinity and satisfies for $a > 0$

$$\int_a^\infty \frac{\gamma(x)}{x^2}\,dx = \infty.$$

The conditions for the quasi-analyticity of the function (50) were later obtained in another form by B. S. Pavlov /78/, who showed that the requirement (49) cannot be significantly weakened. He established that not only condition (48) by itself but even an infinite sequence of conditions

$$\int_0^\infty x^n |q(x)|\,dx < \infty \qquad (n = 0, 1, 2, \ldots) \tag{51}$$

does not yet imply the finiteness of the set of nonpositive points of the spectrum of the operator L_h. Under conditions (51), the function (50) is indefinitely differentiable on the real axis, but may not belong to a quasi-analytic class. B. S. Pavlov's proof consists in the construction of a suitable example. Using the results obtained in the inverse problems of spectral analysis and scattering theory (see /28/, /67/), it is possible to find such an example in the class of operators with a real potential $q(x)$ and a complex parameter h. Moreover, B. S. Pavlov has established that there are in this class operators whose nonreal spectrum clusters to some perfect set of points of the semi-axis $\lambda \geqslant 0$ having null linear measure. Under condition (48), it is impossible that the set of cluster points of the nonreal eigenvalues be of positive measure, but in the class of minimal symmetric operators L with the spectrum $S(L) = [0, \infty)$ there exist operators admitting a nonself-adjoint extension whose nonreal discrete spectrum is contracted to a certain segment of the axis $\lambda \geqslant 0$.

A number of results regarding the spectrum of the Schroedinger operator (5) in $\mathfrak{S}_2(\mathfrak{H}_m)$ with a complex potential were obtained by R. M. Martirosyan /66/. In particular, for $m = 3$ he established that when the function $q(x)e^{\varepsilon|x|}$ is bounded for some $\varepsilon > 0$, the set of points of the spectrum of the operator L that lie outside $C(L) = [0, \infty)$ is finite. It is empty if $\sup_{x \in \mathfrak{H}} |q(x)| e^{\varepsilon|x|} < \frac{\varepsilon^2}{4}$, and in this case the continuous part of the spectrum $C(L)$ contains no eigenvalues /66 (2)/.

67. The model of the crystal lattice. The simplest Born-Von Kármán model of such a lattice is a one-dimensional infinite chain of atoms. The one-dimensionality of the chain implies that the atomic oscillations are polarized. The inter-atomic forces decrease so rapidly with distance that only nearest neighbor interaction need be taken into account. Besides the interaction forces between the atoms, each atom is bonded to its equilibrium position by an elastic force. In a sense, the equations of motion in this model do not differ from those for beads on an elastic string and have the form

$$\frac{d^2 y_k}{dt^2} = \alpha_{k-1} y_{k-1} + \beta_k y_k + \alpha_k y_{k+1} \qquad (k = \pm 0, 1, 2, \ldots).$$

If the lattice spacing is equal to $n\,(1 \leqslant n < \infty)$, then $\alpha_{k+n} = \alpha_k$, $\beta_{k+n} = \beta_k$ and finding the eigenfrequencies reduces to the spectral analysis of the periodical Jacobi matrix T having the elements

$$t_{kk} = \beta_k, \quad t_{k,k+1} = t_{k+1,k} = \alpha_k \qquad (k = \pm 0, 1, 2, \ldots).$$

Periodical as well as limitingly-periodical Jacobi matrices and their associated continuous fractions were for a long time the object of study for various authors (see /30(1)/ and the bibliography given there).

In this subsection we give an account of an algebraic approach to such matrices. Together with the methods of the theory of operators it leads

to some new results concerning the character of the discrete spectrum introduced in the gap of $S(T)$ by completely continuous perturbations of the matrix T. The results given below are due P. B. Naiman /71(2)/.

An infinite matrix A with complex elements a_{jk} is called a generalized Jacobi matrix of order $2m$, if $a_{jk}=0$ for $|j-k|>m$, but $a_{jk}\neq 0$ for $|j-k|=m$ ($j, k=\pm 0, 1, 2, \ldots$). Thus, the ordinary Jacobi matrices are generalized Jacobi matrices of the second order.

The starting point of the investigation is the following discrete analog of the theorem of Barchnell and Chendi on linear differential expressions (see /3/).

Theorem 3 /71(2)/. If the generalized Jacobi matrices A and B of orders $2m$ and $2n$ respectively are permutable, they satisfy an algebraic relation

$$D(A, B)=0$$

of degree $2n$ with respect to A and degree $2m$ with respect to B.

We introduce a special generalized matrix E_n of order $2n$, all the elements of which are equal to zero, with the exception of the elements $e_{k, k-n}=e_{k, k+n}=1$.

If a generalized Jacobi matrix A has the period n, it obviously commutes with E_n, so that

$$D(A, E_n)=0. \tag{52}$$

In the simplest case, for an ordinary periodical symmetric Jacobi matrix of period n (henceforth called "n-periodical") with real elements, the polynomial $D(\lambda, \mu)$ in (52) is a complete square, so that this matrix satisfies the equation

$$P(T)=E_n, \tag{53}$$

where $P(\lambda)$ is a polynomial of degree n. Since the spectrum $S(E_n)$ is then the segment $-2\leqslant\mu\leqslant 2$ and each point $\mu\in S(E_n)$ is of multiplicity $2n$, and since the multiplicity of the spectrum $S(A)$ does not exceed 2, from relation (52) we have the following theorem.

Theorem 4 /71(2)/. The spectrum of a real symmetric periodical Jacobi matrix T coincides with the complete inverse image of the segment $-2\leqslant\mu\leqslant 2$ under the mapping determined by the algebraic equation

$$P(\lambda)=\mu,$$

all whose roots are real for $-2\leqslant\mu\leqslant 2$. In this case, each point of $S(T)$ is of multiplicity 2.

It follows from Theorem 4 that the spectrum $S(T)$ is a system of n segments, which in particular cases may have common points. These segments coincide with the stability zones of the corresponding finite-difference equation, which are defined similarly to the zones (62) of No. 56.

It is similarly established that in the general case of a nonsymmetric periodic matrix T with complex elements the spectrum $S(T)$ is the complete inverse image of some ellipse of the μ-plane under the mapping (54). In this case, $S(T)$ is a system of n algebraic arcs in the complex λ-plane (curvilinear stability zones), containing not more than n connected components.

Let us enumerate from left to right the gaps in the spectrum $S(T)$ of the real self-adjoint n-periodical Jacobi matrix T with the indices $0, 1, 2, \ldots, n$ if n is even, and with the indices $1, 2, \ldots, n+1$ if n is odd. We start with

the gap from $-\infty$, and end with the gap going to $+\infty$. Let the matrix T be perturbed by adding a diagonal real matrix K whose elements k_r tend to 0 for $|r|\to\infty$. Since the perturbation K is completely continuous, the spectrum $S(T+K)$ consists of the system of segments which compose the spectrum $S(T)$ and of a bounded set of eigenvalues which can cluster only at the ends of the gaps in $S(T)$. The corresponding perturbation of the matrix $P(T)$,

$$Q=P(T+K)-P(T),$$

is a generalized Jacobi completely continuous matrix of order $2n-2$. Thus, the polynomial $P(\lambda)$ transforms the limitingly-periodical matrix $T+K$ into the limitingly-constant matrix

$$P(T+K)=E_n+Q. \tag{55}$$

Now, using the methods of No. 37, it is easy to find sufficient conditions for the finiteness and the infiniteness of the set of points of the spectrum $S(E_n+Q)$ that are situated to the left of the point $\mu=-2$, or to the right of the point $\mu=2$. By virtue of relation (55) these conditions coincide with the conditions for the finiteness or the infiniteness of the set of points of the spectrum $S(T+K)$ in the join of all the odd or all the even gaps of the spectrum $S(T)$.

For brevity of the formulation of the results obtained in this way, we confine ourselves to matrices T with period $n=2$ and set

$$a_r=\frac{1-(-1)^r}{2}a_1+\frac{1+(-1)^r}{2}a_2.$$

To be definite we assume that $a_1a_2>0$ and $\beta_2-\beta_1>0$. Furthermore, we introduce the quantities

$$\omega_r'=\frac{|k_r+k_{r-1}|}{|a_r|}+\frac{|k_r+k_{r+1}|}{|a_{r+1}|}-q_r,$$

$$\omega_r''=\frac{|k_r+k_{r-1}|}{|a_r|}+\frac{|k_r+k_{r+1}|}{|a_{r+1}|}+q_r,$$

where

$$q_r=\frac{(-1)^r(\beta_2-\beta_1)k_r+k_r^2}{a_1a_2}.$$

The second gap in $S(T)$ is called the internal, and the join of the first and third gaps is called the external gap of $S(T)$. Let S'' denote the part of the spectrum of the perturbed operator $T+K$ lying in the internal gap and let S' denote the part of the spectrum of this operator situated in the external gap. We have the following theorem.

Theorem 5 /71(2)/. If

$$\lim_{|r|\to\infty}\sup r^2\omega_r'<1, \tag{56}$$

then S'' is finite. If

$$\lim_{|r|\to\infty}\sup r^2\omega_r''<1, \tag{57}$$

then S' is finite. If one of the four series

$$\sum_{r=0}^{\pm\infty}q_{2r},\quad \sum_{r=0}^{\pm\infty}q_{2r+1} \tag{58}$$

diverges to $-\infty$, then S'' is infinite. If one of these series diverges to ∞, then S' is infinite.

From Theorem 5 it follows, in particular, that if for $|r| \to \infty$ we have $k_r = o\left(\frac{1}{r^2}\right)$, then the perturbation K may introduce only a finite number of eigenvalues in the gaps of the spectrum $S(T)$. On the other hand, if one of the two series $\sum^{\pm\infty} k_{2r-1}$ consists only of positive terms and diverges, the perturbation K introduces an infinite set of eigenvalues in the interval gap. Similarly, if one of the series $\sum^{\pm\infty} k_{2r}$ consists only of positive terms and diverges, the perturbation K introduces an infinite set of values in the external gap.

Using Theorem 7[bis] of No. 3 dealing with one-sided perturbations, it is possible, in cases when almost all the k_r have the same sign, to discover to which of the ends of the gap the eigenvalues cluster if they form an infinite set.

In the general case of any period n similar results are obtained. One has then to set $\omega'_r = \sigma_r - q_r$, $\omega''_r = \sigma_r + q_r$ in the left-hand sides of inequalities (56) and (57), where σ_r is the sum of the absolute values of all the elements of the r-th line of the matrix Q, except its diagonal element q_r, and to replace in the right-hand sides of these inequalities the constant 1 by $n^2/4$. Thus, the condition $k_r = o\left(\frac{1}{r^2}\right)$ is still sufficient for the finiteness of the set of points of the spectrum of the operator $T+K$ in the join of all gaps. In order that the set of points $S(T+K)$ that lie in the join of all the odd or even gaps be infinite, it is sufficient to require the divergence of one of the $2n$ series $\sum_{r=0}^{\pm\infty} q_{j+nr}$ $(j = 1, 2, \ldots, n)$ to $\pm\infty$.

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EXPLANATORY LIST OF ABBREVIATIONS OF SOVIET INSTITUTIONS AND JOURNALS APPEARING IN THE REFERENCES

Abbreviation	Full name (transliterated)	Translation
AN UkrSSR	Akademiya Nauk Ukrainskoi SSR	Academy of Sciences of the Ukrainian SSR
DAN SSSR	Doklady Akademii Nauk SSSR	Reports of the Academy of Sciences of the USSR
LGU	Leningradskii gosudarstvennyi universitet	Leningrad State University
MVO	Ministerstvo vysshego obrazovaniya	Ministry of Higher Education

Absolutely continuous part of the operator, A_c, 9
— — — spectrum, $S(A_c)$, 9

Capacity of a set, 152
Characteristic exponent, 188
— of the operator, 76
— sequence of elements for $\lambda \in C(T)$, 76
— — — $\lambda = -\infty$, 66
Classes of functions, 27
Conjugate solutions, 140
Concentration spectrum, 7
Continuous component of the singular part of the spectrum, 9
— spectrum (continuous part of a spectrum) $C(T)$, 7

Decomposition, 10
— point, 28
— surface Γ, 59
Defect number of a J-symmetric operator, 77
— — of a subspace, 11
Direct sum, 8
Discrete component of the singular part of the spectrum, 9
— spectrum (discrete part of a spectrum), 6
Discreteness of the spectrum to the left of a given point, 15
Dissipative operator, 74
Domain of definition of an operator, 6

Eigenvalue rank, 7
Equivalent boundary condition at infinity, 52
Euclidian space, 47
External defect number Def T, 76
— regularity field $R(T)$, 76

Finite characteristic of an operator, 76
— deformation of a domain, 69
— -dimensional decomposition, 10
— — extension, 10
Finite (N-finite) function, 27, 68
— — vector, 44, 126
Function (which is) finite in a given domain, 47

Gap of the continuous part of the spectrum, 14
(Generalized) Jacobi matrix, 217

Induced boundary-value problem, 153
Infinite-dimensional extension, 10
Instability zone, 189

J-self-adjoint operator, 76
J-symmetric operator, 76

Limit-cylindrical domain, 167
— spectrum, 7
Lower bound of an operator, 17

Metric (A- and $A \sim$ metrics), 18
Minimal unclosed differential operator, 28
Minimal differential operator, 28, 47
Monotonic increasing in the main, 33
Multiplicity of an eigenvalue, 6
Multiplicator, 188

Nodal contour, 158
Non-essential cut of a set, 152
Non-negative operator, 17
Non-oscillatory operation, 45, 159
— equation, 40, 44
Normalized sequence, 7

Operator-$\tilde{\Delta}$, 167
Orthogonal complement, 8
— sum, 8
Oscillatory equation (operation), 40, 44, 45, 158

Polar differential operation, 93
Potential, 47, 82
— function, 47
Positive operator, 17
Proper subspace, 6

Quasi-bounded domain, 160
— -conical domain, 159
— -differential operation, 27

Rank of an eigenvalue, 7
Regular differential operation, 47
— — operator, 51
— (D-regular) boundary-value conditions, 51, 63, 67
— -type point, 6
Regularity field $\Pi(T)$, 6
— point of an operator, 6
Relation $\omega_1 \subset \omega_2$, 48
Relative complete continuity, 18
Residual (part of the) spectrum, 7
Resolvent set, 6
Root manifold, 6
— subspace, 7

Semi-bounded operator, 17
Separable root subspace, 7
Separated boundary conditions, 51
Set $K(T)$, 75
Singular boundary-value conditions, 51
— differential operation, 47
— — operator, 51
— part of the operator, 9
— — of the spectrum, 9
Spectrum, 6
Stability zone, 189
Stable (h-stable) property of the spectrum, 82
— value of the parameter λ, 188
Strict extension $\mathbf{A}$, 17
Support of a finite function, 27, 47

Two-term differential operation, 82
Type of each point of the continuous part of the spectrum, 14

Unstable value of the parameter λ, 188